Supplement

→ Deriving MHD

→ MHD is derived from 2-fluid equations

- first discuss 2 fluid derivation from Boltzmann
- then, discuss reduction to one-fluid MHD (i.e. approximations/limitations — especially in Ohm's Law)

→ deriving fluid equations

Have in general, Boltzmann eqn

$$\underbrace{\frac{\partial f}{\partial t}}_{①} + \underbrace{\underline{v}\cdot\underline{\nabla} f}_{②} + \underbrace{\frac{q}{m}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\underline{\nabla} f}_{③} = \underbrace{C(f)}_{④}$$

$C(f)$ → collision operator

and can assign time scale

① ↔ ω → frequency

② ↔ $v_{Th}/L_{\parallel}$
↳ relevant parallel scale

③ $\frac{q}{m}\frac{E}{\Delta V}$ $\Delta V \sim V_{Th} \to$ non-resonant

$\Delta V \sim \Delta V_{Tr} \to$ resonant

NL scattering rate (→ small, usually)

④ ν_{eff} - collision frequency

For "fluid description", need:

→ $\nu_{eff} > V_{Th}/L_{\parallel}$

i.e short mean free path limit

or

→ $\omega > V_{Th}/L_{\parallel}$ → ala' gyrokinetic KSAW, where $\nu \to 0$

i.e. "fluid" ↔ blob / fluid element of particles

→ what holds blob together? (i.e. prevents dispersal?)

⇒ collisions (i.e. particles collide and scatter prior dispersal)

or ⇒ vibrations in wave.

here, focus on short mean-free path ordering.

For $C(f) \gg \partial f/\partial t$, $\underline{v}\cdot\nabla f$, etc.

l.O. $\quad C(f) = 0$

$$\Rightarrow f = f_{\text{Maxwellian}}$$

i.e. – collisions drive distribution function to local Maxwellian on time scale short compared all else

– n.b. Maxwellian can be shifted, and have gradients.

1st order:

$$\frac{\partial f^{(0)}}{\partial t} + \underline{v}\cdot\nabla f^{(0)} + \frac{q}{m}\left(\underline{E} + \frac{\underline{v}}{c}\times\underline{B}\right)\cdot\underline{\nabla} f^{(0)} = C(f^{(1)})$$

then integrating:

$$\int d^3v \left[\frac{\partial f^{(0)}}{\partial t} + \nabla\cdot \underline{v} f^{(0)} + \overbrace{\frac{\partial}{\partial \underline{v}}\cdot\left(\frac{q}{m}\left(\underline{E} + \frac{\underline{v}}{c}\times\underline{B}\right) f^{(0)}\right)}^{\text{IBP} \atop 0}\right]$$

$$= \int d^3v \; C(f^{(1)})$$

Now, $\int d^3v \, C(f_\cdot) = 0$ → collisions conserve #!

so, have:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{V}) = 0$$

i.e. continuity equation.

$$\begin{cases} n = \int d^3v \; f \\ \underline{V} \equiv \int d^3v \; \underline{v} f \Big/ n \end{cases}$$

→ basic moments.

→ Now first order moment:

$$\int d^3v\, \underline{v} \left(m\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f m + q\left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot \underline{\nabla}_v f = C(f) \right)$$

(1) (2) (3) (4)

$$(1) = m \frac{\partial}{\partial t}\left(n \underline{V}\right) \qquad \underline{V} = V(\underline{x}, t)$$

$$(3) = \int \underline{v} q\left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot\frac{\partial f}{\partial \underline{v}}$$

$$= \int \frac{\partial}{\partial \underline{v}}\left[f\underline{v}\left(\underline{E} + \underline{v}\times\underline{B}\right)\right] d^3v - \int f\underline{v}\, \frac{\partial}{\partial \underline{v}}\cdot\left(\underline{E} + \underline{v}\times\underline{B}\right)$$

0, 0, $\perp \underline{v}$

$$- \int f\left(\underline{E} + \underline{v}\times\underline{B}\right)\cdot\frac{\partial \underline{v}}{\partial \underline{v}}$$

so

$$= -qn\left(\underline{E} + \underline{V}\times\underline{B}\right)$$

$$(4) = \int d^3v\; m C(f)\, \underline{v}$$

$$= \underline{P}_{ij}$$

→ collisional momentum transfer from species i to j which leaves ②:

$$② = m\int d^3v \ \underline{v}(\underline{v}\cdot\underline{\nabla}) f$$

$$= m\int d^3v \ \underline{\nabla}\cdot(f\underline{v}\,\underline{v})$$

$$= \nabla\cdot\left[m\int d^3v \, f\,\underline{v}\,\underline{v}\right] = m\,\underline{\nabla}\cdot(n\,\overline{\underline{v}\,\underline{v}})$$

clearly useful to separate $\underline{v}$ into mean and fluctuating pieces

$$\underline{v} = \underline{V} + \underline{w}$$

$$\Rightarrow \underline{\nabla}\cdot(n\,\overline{\underline{v}\,\underline{v}}) = \underline{\nabla}\cdot(n\underline{V}\,\underline{V}) + \underline{\nabla}\cdot(n\,\overline{\underline{w}\,\underline{w}})$$

$$+ \underline{\nabla}\cdot n\,(\underbrace{\underline{V}\,\overline{\underline{w}} + \overline{\underline{w}}\,\underline{V}}_{0,\ \text{defn.}})$$

$$\underline{\nabla}\cdot(n\underline{V}\,\underline{V}) = \underline{V}\ \underline{\nabla}\cdot(n\underline{V}) + n\,(\underline{V}\cdot\underline{\nabla})\,\underline{V}$$

$$n\,\overline{\underline{w}\,\underline{w}} \equiv \underline{p}$$

pressure tensor !.

so, can write for momentum equation

$$m\frac{\partial}{\partial t}(n\underline{V}) + m\underline{V}\,\underline{\nabla}\cdot(n\underline{V}) + mn(\underline{V}\cdot\underline{\nabla})\underline{V}$$
$$+ \underline{\nabla}\cdot\underline{P} - qn(\underline{E} + \underline{V}\times\underline{B}) = \underline{R}_{ij}$$

and using continuity:

$$\boxed{mn\left[\frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn(\underline{E} + \underline{V}\times\underline{B}) - \underline{\nabla}\cdot\underline{\underline{P}} + \underline{R}_{ij}}$$

Now, for form $\underline{P}$:

$$\underline{\underline{P}} = \int d^3v\, m\, v_i v_j f$$

in short mean-free-path ordering,

$$f \cong f_{Maxwellian}$$

As mean extracted, symmetry $\Rightarrow$

$$\underline{\underline{P}} = \int d^3v\, v_i v_j \delta_{ij}\, f$$

so $\underline{\underline{P}} = \begin{bmatrix} P_1 & & 0 \\ 0 & P_2 & \\ & & P_3 \end{bmatrix}$ pressure tensor diagonal

if isotropic : $P_1 = P_2 = P_3$ (fast $\parallel, \perp$ thermal equilibration)

$\Rightarrow$

$$\underline{\underline{P}} = p\,\underline{\underline{I}}$$

and pressure reduces to scalar, i.e.

$$mn\left[\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right] = qn\left(\underline{E} + \underline{V}\times\underline{B}\right) - \underline{\nabla} p + \underline{P}_{ij}$$

$\Rightarrow$ For second order moment $\rightarrow$ energy (closure $\Leftrightarrow$ energy flux!) $\Rightarrow$ eqn. state

2 species $\Rightarrow$ $p/\rho^{\gamma} = \text{const.}$

→ Single Fluid (→ MHD)

Can define single fluid variables:

$$\rho = n_i M + n_e M \cong n M$$ → density

mass velocity:

mean velocity

$$\underline{V} = \frac{1}{\rho}\left(n_i M \underline{V}_i + n_e M_e \underline{V}_e\right)$$

$$\approx \left[\frac{M \underline{V}_i + m_e \underline{V}_e}{M+m}\right] \cong \underline{V}_i$$

relative velocity

Current density:

$$\underline{J} = q\,(n_i \underline{V}_i - n_e \underline{V}_e)$$

$$\approx nq\,(\underline{V}_i - \underline{V}_e)$$, using QN.

Upshot:

- continuity for ions ⇒ single fluid continuity

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{V}) = 0}$$

- adding electron and ion momentum eqns:

$$Mn\left(\frac{\partial \underline{V}_i}{\partial t} + \underline{V}_i \cdot \underline{\nabla}\, \underline{V}_i\right) = qn\left(\underline{E} + \underline{V}_i \times \underline{B}\right) - \underline{\nabla} p_i + \underline{P}_{i,e}$$

$$\oplus$$

$$m_e n\left(\frac{\partial \underline{V}_e}{\partial t} + \underline{V}_e \cdot \underline{\nabla}\, \underline{V}_e\right) = -qn\left(\underline{E} + \underline{V}_e \times \underline{B}\right) - \underline{\nabla} p_e + \underline{P}_{e,i}$$

$\Rightarrow$

$$n\left(\frac{\partial}{\partial t}\left(M\underline{V}_i + m\underline{V}_e\right) + M(\underline{V}_i \cdot \underline{\nabla})\underline{V}_i + m_e(\underline{V}_e \cdot \underline{\nabla})\underline{V}_e\right)$$

$$= qn\left(\underline{V}_i - \underline{V}_e\right) \times \underline{B} - \underline{\nabla}(p_i + p_e)$$

$$+ \underbrace{\cancel{\underline{P}_{e,i} + \underline{P}_{i,e}}}_{\text{momentum cons.}}$$

as: $m_e \ll M$

$\underline{V}$ defn.

$\rho = \rho_e + \rho_i$

$$\Rightarrow \boxed{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V}\right) = \underline{J}\times\underline{B} - \underline{\nabla}p + \underset{\substack{\downarrow \\ \text{any additional} \\ \text{body force.}}}{F_{body}}}$$

Momentum balance.

$\Rightarrow$ Now, only re-maining non-trivial MHD equation is Ohm's Law.

$\rightarrow$ where the bodies are buried, . . .

Consider, [m_e * (ion momentum eqn) —

M * (electron momentum eqn.)]

$$\Rightarrow M m_e n\left(\frac{\partial}{\partial t}(\underline{V}_i - \underline{V}_e) + \underline{V}_i\cdot\underline{\nabla}\,V_i - \underline{V}_e\cdot\underline{\nabla}\,\underline{V}_e\right)$$

$$= qn\,(M+m_e)\,\underline{E} + qn\,(m\underline{V}_i + M\underline{V}_e)\times\underline{B}$$

$$- m\nabla p_i + M\nabla p_e - (M+m)\,\underline{P}_{ei}$$

Now, ① $\underline{P}_{ei}$ = electron-ion momentum transfer

$$= -M n q \eta \underline{J}$$

② $M >> m_e$

③ neglecting advective derivatives

$\Rightarrow$

$$\frac{M m_e n}{q} \frac{\partial}{\partial t}\left(\frac{\underline{J}}{n}\right) = q\rho \underline{E} - M n q \eta \underline{J} + M \nabla p_e + qn(m\underline{V}_i + M\underline{V}_e)\times\underline{B}$$

and can further simplify:

$$m\underline{V}_i + M\underline{V}_e = M\underline{V}_i + m\underline{V}_e - (M-m)(\underline{V}_i - \underline{V}_e)$$

$$\cong \frac{\rho \underline{V}}{n} - \frac{M}{nq}\underline{J}$$

Finally, re-arrangeing $\Rightarrow$

$$\boxed{\frac{m_e}{nq^2}\frac{\partial \underline{J}}{\partial t} = \left(\underline{E} + \frac{\underline{V}\times\underline{B}}{c}\right) - \eta \underline{J} - \frac{1}{nq}(\underline{J}\times\underline{B}) + \frac{1}{nq}\nabla p_e}$$

Now, have generalized Ohm's Law:

$$\underset{①}{\frac{m_e}{nq^2}\frac{\partial \underline{J}}{\partial t}} = \underset{②}{\left(\underline{E} + \frac{\underline{V}\times\underline{B}}{c}\right)} - \underset{③}{\eta \underline{J}} - \underset{④}{\frac{\left(\underline{J}\times\underline{B}\right)}{nq}} + \underset{⑤}{\frac{\underline{\nabla} P_e}{nq}}$$

② ⇒ ideal MHD Ohm's Law
③ ⇒ collisional resistivity

} resistive MHD

'ring on ④ : Hall Term

⇒ Hall MHD

bring in ⑤ : Electron thermal force / pressure

⇒ diamagnetic / finite electron ω_* MHD

i.e. Boltzmann response : $\underline{E}$ vs $\frac{\underline{\nabla} P_e}{nq}$

① : Electron inertia term ($\sim m_e$)

⇒ EMHD, electron inertially modified MHD.
($\omega\, m_e/nq^2 > \eta$)

For low frequency, strong collisionality, etc.

$$\Rightarrow \quad \underline{E} + \underline{V}\times\underline{B} = \eta\,\underline{J} \quad \Big\} \text{ Resistive MHD.}$$

N.B. :- Ohm's Law is most sensitive part of MHD structure → need care.

- high ω → electron inertia
  tok. μ-inst → thermal force term.
  $\lambda \sim c^2/\omega_{pi}^2$ → Hall term.

$$\boxed{\underline{V}_d = \frac{1}{2}\frac{V_\perp^{3}}{\Omega}\frac{\underline{B}\times\nabla B}{B^2}} \rightarrow \nabla B \text{ drift.}$$

↳ particles drift ⊥ to field and direction of field variation

→ opposite for e, i → → current develop
→ rotational transform
short out

Note: total drift due magnetic inhomog.

$$\boxed{\underline{V}_{\perp d} = \frac{V_\parallel^{2}}{\Omega}\frac{\underline{R}_c\times\underline{B}}{B R_c^2} + \frac{V_\perp^{3}}{2\Omega}\frac{\underline{B}\times\nabla B}{B^2}}$$

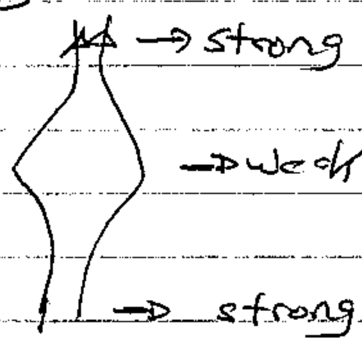
- ExB, - pol > φ
- ∇B, - curv >

— Left: Drifts DKE
→ — Adi Inv
— GKE

(iii.) Adiabatic Invariants
→ useful in curved field

{ B – moment
bounce circ.
drift }

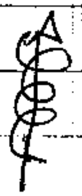
→ first, what of field intensification along B



Mirror
RIP

obviously magnetic mirroring will occur

→ to describe, convenient to introduce concept of magnetic moment



$$\mu = (\text{current}) \times (\text{area}) / c \qquad \text{before}$$

$$= \left(\frac{q\Omega}{2\pi}\right)\left(\frac{\pi \rho_L^2}{c}\right)$$

$$\mu = \frac{m v_\perp^2}{2B}$$

Ignoring $\underline{E} \times \underline{B}$ drift (i.e. $\underline{E} = 0$, at g.c.),

$$\frac{d}{dt}\left(\frac{m v_\perp^2}{2}\right) = q\,\underline{E} \cdot \underline{v}_\perp$$

then, averaging over 1 cyclotron orbit:

$$\delta\left(\frac{m v_\perp^2}{2}\right) = \int_{\Omega^{-1}} dt\, q\,\underline{E} \cdot \underline{v}$$

$$= \oint_{\rho} d\underline{\ell} \cdot \underline{E}\, q$$

$$= \int_{\rho} d\underline{a} \cdot q\, \nabla \times \underline{E}$$

$$= \int_{\rho} d\underline{a} \cdot \left(\frac{+q}{c}\frac{\partial \underline{B}}{\partial t}\right)$$

~~sign → contour follows particle motion~~

$$= \pi \rho_L^2 \left(\frac{+q}{c}\right) \frac{\partial B}{\partial t}$$

$$\therefore \quad \delta\left(\frac{mv_\perp^2}{2}\right) = +\pi \frac{q}{c} \frac{v_\perp^2}{\frac{q^2B^2}{m^2c^2}} \frac{\partial B}{\partial t}$$

change in cyclotron period

$$= +\frac{mv_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but, $\delta B = +\frac{2\pi}{\Omega} \frac{\partial B}{\partial t} \equiv \delta t \frac{\partial B}{\partial t}$

change in one period

$$\Rightarrow \quad \delta\left(\frac{mv_\perp^2}{2}\right) = +\frac{mv_\perp^2}{2} \frac{1}{B} \frac{\partial B}{\partial t}$$

$$\therefore \quad \boxed{\delta\left(\frac{mv_\perp^2}{2B}\right) = 0}$$

$\rightarrow$ magnetic moment invariant on $t \gg \Omega^{-1}$ (adiabatic inv.)

Now:

$\rightarrow$ obviously μ not invariant on time scales Ω^{-1} $\leftrightarrow$ averaging!

$\rightarrow$ re: mirroring, if $B(\ell) \gg B(0)$

$$\Rightarrow v_\perp^2(\ell) \gg v_\perp^2(0)$$

$$\therefore v_\parallel^2(\ell) \ll v_\parallel^2(0)$$

$$V_\parallel^2(0) + V_\perp^2(0) = V_\parallel^2(\ell) + V_\perp^2(\ell)$$

(energy)

$$\frac{m V_\perp^2(0)}{2B(0)} = \frac{m V_\perp^2(\ell)}{2B(\ell)}$$

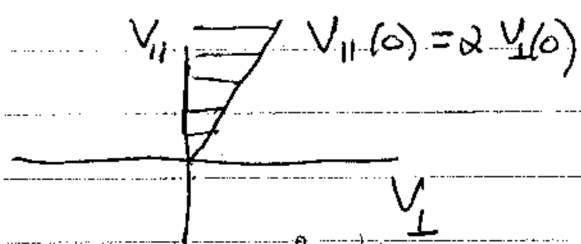
$$\Rightarrow \quad V_\perp^2(\ell) = \frac{B(\ell)}{B(0)} V_\perp^2(0)$$

$$\Rightarrow \quad V_\perp^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) + V_\parallel^2(0) = V_\parallel^2(\ell)$$

Confinement $\Rightarrow V_\parallel^2(\ell) = 0$

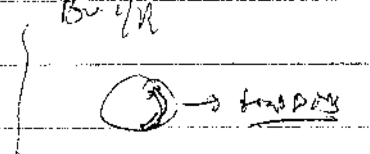
$\rightarrow$ loss cone

$$\Rightarrow \quad \frac{V_\parallel^2(0)}{V_\perp^2(0)} \leq \left(\frac{B(\ell)}{B(0)} - 1\right)$$



$\rightarrow$ mirror ratio

$$\alpha = \left(\frac{B(\ell)^2}{B(0)^2} - 1\right)^{1/2}$$



i.e.

$\rightarrow$ some class of particles always lost

$\rightarrow$ distribution of survivors has loss cone 'hole' in it.

$$\rightarrow \quad V_\parallel = \left[\frac{2}{m}(E - \mu B)\right]^{1/2}$$

$\rightarrow$ to 14

Adiabatic Invariants, cont'd (skip)

As adiabatic invariants, such as μ, etc., clearly are quite useful, it's worthwhile to develop general theory.

Thus consider:

- finite 1D motion
- system/external field characterized by $\lambda(t)$ parameter s/t

$$\underbrace{\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \omega}_{}$$

$$\omega \rightarrow \text{freq.}$$

$\therefore$ - E not conserved, but

- $\dot{E} \sim \dot{\lambda}$ $\Rightarrow$ suggests some (linear) combination E, λ s/t combo invariant

Now, consider $H = H(p, q, \lambda)$

$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}$$

as λ varies slowly, by construction

$$\frac{d\bar{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}}\frac{d\lambda}{dt}$$

where $\overline{} \equiv \frac{1}{\tau}\int_0^{\tau} dt \qquad \tau = \frac{2\pi}{\omega}$

Noting $\tau = \oint dq \big/ dq/dt$ path for given particular λ

$$= \oint dq \big/ (\partial H/\partial p)$$

$\Rightarrow$

$$\frac{d\bar{E}}{dt} = \frac{d\lambda}{dt} \frac{\oint (\partial H/\partial\lambda)\, dq \big/ (\partial H/\partial p)}{\oint dq \big/ (\partial H/\partial p)}$$

where path at fixed λ.

Now, at such at path, H fixed and $\underset{\sim}{E \text{ constant}}$. Thus, on such a path

$$p = p(q; \underbrace{E, \lambda}_{\text{indep, const param}})$$

and $\frac{d}{d\lambda}\left(H(p, q; \lambda) = E\right)$

$$\Rightarrow \quad \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0$$

so $\dfrac{\partial H/\partial \lambda}{\partial H/\partial p} = -\dfrac{\partial p}{\partial \lambda}$

and plugging into $\dot{E}$ $\Rightarrow$

$$\frac{dE}{dt} = \frac{d\lambda}{dt}\,\frac{\oint dq\,(-\partial p/\partial \lambda)}{\oint dq\,(\partial p/\partial E)}$$

$\Rightarrow$

$$\oint dq\left(\frac{\partial p}{\partial E}\frac{dE}{dt} + \frac{\partial p}{\partial \lambda}\frac{d\lambda}{dt}\right) = 0$$

for $p = p(E, \lambda)$ $\Rightarrow$

$$\frac{d}{dt}\oint dq\, p = 0$$

Thus :- $\underset{(J)}{I} = \oint dq\, p$, for fixed $\binom{E}{\lambda}$

is invariant (corresponds action variable)

- as E, λ (fixed) $\Rightarrow$ I invariant for $T \gg 2\pi/\omega$ s/t

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \omega$$

where recover invariance of linear proportionality.

For waves $\begin{cases} N = \mathcal{E}/\omega \\ \mathcal{E} = \mathcal{E}(\underline{k}, \underline{x}, t) \end{cases}$

is useful adiabatic invariant.

Now, returning to mirror problem. 

→ for μ, first invariant:

$$J = \oint \underline{p} \cdot d\underline{q} \quad , \quad \text{for cyclotron motion}$$

Now, in B field $\quad \underline{p} = m\underline{v} + \frac{q\underline{A}}{c}$

so → canonical momentum

$$J_1 = \oint \left(m\underline{v} + \frac{q\underline{A}}{c} \right) \cdot d\underline{q}$$

now, $\quad A(\underline{r}) = A(\underline{x}_{gc}) + \underline{\rho} \cdot \underline{\nabla} A + \ldots$

as orbit is cyclotron

$$\overset{\text{angle}}{\phi}$$

$$d\underline{q} = \frac{\partial \underline{\rho}}{\partial \alpha} d\alpha$$

$$\underline{\rho} = \frac{\underline{v}_\perp}{\Omega}$$

$$\Rightarrow \quad J_1 = \oint \frac{d\alpha}{\Omega} \, \underline{v}_\perp \cdot \left(m\underline{v}_\perp + \frac{q}{c} \underline{A}(\underline{x}) + \underline{\rho} \cdot \nabla \underline{A} + \cdots \right)$$

n.b. ∥ motion, drifts etc. vanish on integration

$$\therefore \quad J_1 = 2\pi m \frac{v_\perp^2}{\Omega} + 2\pi \frac{q}{c\Omega} \left\langle \underline{v}_\perp \cdot (\underline{\rho} \cdot \nabla \underline{A}) \right\rangle$$

$$= 2\pi \frac{m v_\perp^2}{\Omega} + \frac{2\pi q}{c\Omega} \frac{v_\perp^2}{2\Omega} \underline{b} \cdot \nabla \times \underline{A}$$

$$= 2\pi \frac{mc}{q} \left(\frac{m v_\perp^2}{B} \right) + 2\pi \frac{mc}{q} \left(\frac{m v_\perp^2}{B} \right)$$

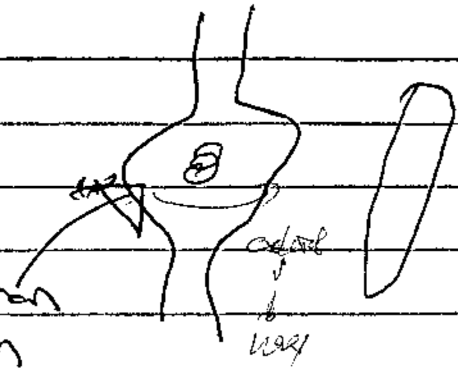
$$\boxed{J_1 = \frac{2\pi mc}{q} \left(\frac{m v_\perp^2}{2B} \right)}$$

$$\therefore \quad J_1 = (\text{const.}) \times \mu$$

recovers J_1

Now, for $\overline{J}_2$:

consider motion



cyclotron motion closed loop

$\Rightarrow$ parallel bounce motion closed loop.

so $\overline{J}_2 = \oint p_{\parallel} \, dS_{\parallel}$

traditional notation.

$$v_{\parallel}^2(0) + v_{\perp}^2(0) = v_{\parallel}^2(\ell) + v_{\perp}^2(\ell)$$

$$\frac{v_{\perp}^2(0)}{B(0)} = \frac{v_{\perp}^2(\ell)}{B(\ell)} = \mu (2m)$$

$$\Rightarrow \quad v_{\parallel}^2(\ell) = v_{\parallel}^2(0) + v_{\perp}^2(0) - \mu B(\ell)$$
$$= 2m\,(E - \mu B(\ell))$$

$$\therefore \quad \boxed{J_2 = \oint d\ell \, \left(2m\,(E - \mu B(\ell))\right)^{1/2}}$$

- 'bounce' invariant

- note $\nabla \cdot B = 0$ forces $B_r \neq 0$

i.e.

$$\nabla_r B_r + \underbrace{\partial_z B_z}_{\neq 0} = 0$$

$\Rightarrow B_r \neq 0$

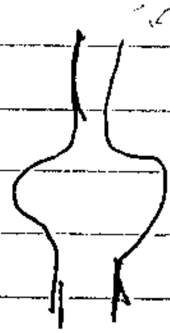
$\therefore$ - J_2 conservation $\Leftrightarrow$ conservation of magnetic flux thru G.C. bounce orbit

flux

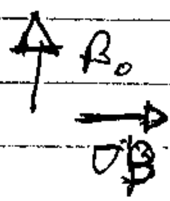
- for motion on field line, conservation of zero!

for J_3: what is 3^{rd} loop?

i.e.



$B_r(r, z) \Rightarrow \nabla B$ drift!



$\Rightarrow$ azimuthal drift around machine / mirror

i.e. $J_3 = \oint \underline{P}_\phi \cdot d\ell$

N.B. time scale hierarchy clear.

→ Systematics ?

- How determine most general adiabatic invariant ?

- clues — additive

i.e. $J = \oint p_{cyc} \cdot dq_{cyc} + \underbrace{\oint p_{\parallel} dl_{\parallel} + \oint \underline{p_d} \cdot \underline{dx_d}}_{\text{G.C.}}$

— time scale hierarchy :

$J_1 \Rightarrow \tau_{cycl.}$
$J_2 \Rightarrow \tau_{bounce}$
$J_3 \Rightarrow \tau_{drift}$

⇒ suggests J_2 corrects J_1, etc.

also ⇒ why care so much ??

Now, most general adiabatic invariant is invariant along particle orbits

i.e. $\dfrac{dI}{dt} = 0$

in absence of E fields

$$\underline{v} \cdot \underline{\nabla} I + \frac{q}{m_c} (\underline{v} \times \underline{B}) \cdot \nabla I = 0$$ (Vlasov eqn.)

$$I = I_0 + \epsilon I_1 + \epsilon^2 I_2 + \ldots$$

Now $-$ $\underline{v} = v_{\parallel} \hat{n} + v_{\perp} \cos\phi \, \underline{e} + v_{\perp} \sin\phi \, \underline{e}'$

$$v_{\parallel} = \left(\frac{2}{m} (E - \mu B) \right)^{1/2}$$

$$v_{\perp} = (2\mu B/m)^{1/2}$$

$-$ take $\underline{B} = \underline{\nabla} \alpha \times \nabla \underline{\beta}$

i.e. $l \rightarrow$ along B

$\alpha, \beta \rightarrow \perp$

Now, can see in these coordinates:

$$\underset{①}{\Omega \frac{\partial I}{\partial \phi}} = \underset{②}{\mathcal{D} I} \qquad \frac{②}{①} \sim O(\epsilon)$$

$\downarrow$

complicated mess

$$I = I_0 + \epsilon I_1 + \epsilon^2 I_2 + \ldots$$

$$\oint d\phi \, \mathcal{D} \equiv \langle \mathcal{D} \rangle = v_{\parallel} \frac{\partial}{\partial l}$$

so $\Rightarrow$ l.o. $\Omega \frac{\partial I_0}{\partial \phi} = 0$

$\Rightarrow I_0 = I_0(\mu, E, \underbrace{\alpha, \beta, l}_{\text{location in field structure}})$ $\Rightarrow$ any fn'tn const on Ω^{-1}

ident. as J l.o. ad. inv

1st order:

$\Omega \frac{\partial I_1}{\partial \phi} = \mathcal{D} I_0$

and $\oint d\phi \Rightarrow$

$\langle \mathcal{D} I_0 \rangle = 0$

$\Rightarrow \quad v_{\parallel} \frac{\partial I_0}{\partial l} = 0$

$\Rightarrow I_0 = I_0(\mu, E, \alpha, \beta)$

and for I_1:

$$\overline{I}_1 = \frac{1}{\Omega}\int^{\phi} d\phi\, \mathcal{D} I_0 + \overline{I}_1(E, \mu, \alpha, \beta, \ell)$$

⇒ Next order:

$$\Omega \frac{\partial I_2}{\partial \phi} = \mathcal{D} I_1$$

$$\Rightarrow \langle \mathcal{D} I_1 \rangle = \langle \mathcal{D} \overline{I}_1 \rangle + \left\langle \mathcal{D} \frac{1}{\Omega}\int \mathcal{D} I_0 \right\rangle = 0$$

evaluting I_0 (messy) ⇒

$$\langle \mathcal{D} I_1 \rangle = v_{\parallel} \frac{\partial}{\partial \ell} \overline{I}_1 + (\underline{v}_d \cdot \nabla I_0) = 0$$

so now can integrate in ℓ such:

$$\boxed{\oint \frac{d\ell}{v_{\parallel}} \underline{v}_d \cdot \nabla I_0 = 0}$$

etc.

⇒ What does above mean?

$\underline{B} = \underline{\nabla}\psi$ s/t $\nabla\alpha \cdot \nabla\psi = \nabla\beta \cdot \nabla\psi = 0$

and

$$\frac{\underline{R_c}}{R_c} = \frac{\nabla_\perp B}{B}$$

plugging in:

$$\oint \frac{d\chi}{B^2}\, \underline{B}\times\nabla\left(\frac{v_\parallel}{B}\right)\cdot\nabla I_0 = 0$$

$$B = \underline{\nabla\alpha}\times\underline{\nabla}\beta$$

$$\underline{\nabla} = \nabla\alpha\frac{\partial}{\alpha} + \underline{\nabla}\beta\frac{\partial}{\partial\beta} + \nabla\chi\frac{\partial}{\partial\chi}$$

$$\Rightarrow \frac{\partial I_0}{\partial\alpha}\frac{\partial}{\partial\beta}J - \frac{\partial I_0}{\partial\beta}\frac{\partial J}{\partial\alpha} = 0 \quad \text{, where}$$

$$J = \oint d\chi\,\frac{v_\parallel}{B} = \oint v_\parallel\, d\ell$$

so $I_0 = I_0(\mu, E, J)$.
$\hookrightarrow J_2$

Now if J_2 exists, $\Rightarrow$

$$J_2 = J_2\,(E, \mu, \alpha, \beta)$$

so for long times, particle must drift on surface described by:

$J(\mu, E, \alpha, \beta) = \text{const.}$

Thus, adiabatic invariants give insight into motion without need to integrate eqns. of motion.

$\Rightarrow$ Now, for $\omega \ll \Omega$ but $k_\perp \rho$ small/finite but non-zero

$\Rightarrow$ gyro-kinetic equation.

i.e.

- modify DKE for finite $k_\perp \rho$
- derive systematically, via time averaging.

Aside

→ Reduced MHD → { Reduced Representation for strong ⊙ straight B_0 — eliminates fast mode

Note: ① full MHD : 3 $\underline{V}$ components
2 $\underline{B}$ " " ($\nabla \cdot \underline{B} = 0$)
ρ, p

⇒ 7 components

② if $\nabla \cdot \underline{V} = 0$ ⇒ 4 components
($\rho = \text{const}$, p from $\nabla \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD
⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:
- assume strong B_z (strong magnetization → gyrokinetics) ↳ later

"strong" ⟷ $\rho V^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in ⊥ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ order : $B_z \sim \nabla_\perp \sim 1$

$B_\perp \sim \partial_z \sim O(\epsilon)$

Take $\rho \sim 1$, as $\underline{\nabla}\cdot\underline{V} = 0$ enforced by strong B_z.

$V_\perp^2 \sim p \sim B_\perp^2$ (i.e. equipartition of energy) (springiness)

$\Rightarrow$ $V_\perp \sim \epsilon$, $p \sim \epsilon^2$, $\partial_t \sim \underline{V}_\perp \cdot \underline{\nabla}_\perp \sim \epsilon$

and pressure balance ($\underline{\nabla}\cdot\underline{V} = 0$ and incompressibility)

$p + B^2 \sim \text{const}$

$\delta(B_z^2) \sim 2B_z \delta(B_z) \sim \delta p$ (eqbm)

i.e. $\omega \ll k\left(c_s^2 + V_A^2\right)^{1/2}$

[idea is to order out the fast mode]

$\Rightarrow \delta B_z \sim \epsilon^2$.

$\therefore$ to lowest order $\Rightarrow B_z = \text{const}$.

Now then: ($\underline{\nabla}\cdot\underline{B} = 0$)

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla}\psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

parallel comp. of vector pot.

$\underline{\nabla}\cdot\underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \to 0$.

Similarly,

$\partial_z p \sim O(\epsilon^3)$, $\Rightarrow$ $V_z \ll V_\perp$
$J_\perp B_\perp \sim \epsilon^3$

$V_z \to 0$

neglect V_z.

Now, $\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = -\frac{\underline{v}\times\underline{B}}{c}$

$$\Rightarrow \quad +\frac{1}{c}\frac{\partial \underline{A}}{\partial t} = \frac{\underline{v}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad (*)$$

$$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{z}$$

so $\partial_t A_\perp \sim \epsilon^3$ (a la $\partial_z \rho_z$)

$$\therefore \quad \nabla_\perp \phi \cong \left(\frac{\underline{v}\times\underline{B}}{c}\right)_\perp, \text{ in } (*)$$

$$\Rightarrow \boxed{\underline{V}_\perp = c\frac{\hat{z}\times\underline{\nabla}\phi}{B_z}}$$

$\perp$ velocity

$\rightarrow$ {motion $\perp$ is $\underline{E}\times\underline{B}$.

Now, taking parallel component of $(*)$.
(units!)

$$\Rightarrow \quad \frac{\partial}{\partial t}\psi + \underline{v}\cdot\underline{\nabla}\psi = B_z\partial_z\phi$$

so have $\binom{\text{vector potential}}{\text{flux}}$ equation:

$$\boxed{\frac{\partial\psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi}$$

$$= B_z \hat{z} + \hat{z}\times\nabla\psi$$

or, alternatively, $$\boxed{\frac{\partial\psi}{\partial t} - \underline{B}\cdot\underline{\nabla}\phi = 0}$$

Finally, for ϕ, write: $\perp$ motion

cells of $E\times B$ drift, ('spin up' rate?)

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{\underline{\nabla} p}{\rho_0} + \frac{\underline{J}\times\underline{B}}{c}$$

$(\underline{\nabla}\times)\cdot\hat{z} \Rightarrow$ vorticity component $(\parallel \hat{z})$ evolution

$$\frac{\partial}{\partial t}\omega_z + \underline{V}_\perp\cdot\underline{\nabla}\,\omega_z = \cancelto{0}{-\underline{\nabla}\times\frac{\underline{\nabla} p}{\rho_0}} + \hat{z}\cdot\underline{\nabla}\times\left(\frac{\underline{J}\times\underline{B}}{c}\right)$$

$$= \underline{B}\cdot\underline{\nabla}J_z - \cancel{\underline{J}\cdot\underline{\nabla}B_z} \qquad \delta B_z \sim \epsilon^3$$

$$\cong \underline{B}\cdot\underline{\nabla}J_z$$

$\therefore$ $$\boxed{\frac{\partial}{\partial t}\omega_z + \underline{V}\cdot\nabla\,\omega_z = \underline{B}\cdot\underline{\nabla}J}$$

but:

$$\omega_z = \hat{z}\cdot\underline{\nabla}\times\underline{V} = \nabla^2\phi$$

$$J_z = \hat{z}\cdot(\underline{\nabla}\times\underline{B})\frac{c}{4\pi} = \nabla^2\psi$$

so finally have:

$$\boxed{\frac{\partial}{\partial t}\nabla^2\phi + \underline{v}\cdot\underline{\nabla}\,\nabla^2\phi = B_z\frac{\partial}{\partial z}\nabla^2\psi + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi}$$

Finally have reduced MHD equation:

$$\boxed{\begin{aligned}&\frac{\partial\psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi \\ &\frac{\partial}{\partial t}\nabla^2\phi + \underline{v}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi \\ &\qquad = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi\end{aligned}}$$

- note have reduced MHD to 2 scalar evolution equations

- does this look familiar?

even stronger!

– for 2D MHD:

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = \underline{B}\cdot\underline{\nabla}\,\nabla^2\psi + \nu\nabla^2\nabla^2\phi$$

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = \eta\nabla^2\psi$$

– ① Conservation Laws, etc. (HW)

$$\frac{d}{dt}E \underset{(\text{to } \eta,\ \nu)}{=} 0, \quad E = \int d^3x \left[\frac{(\nabla\phi)^2}{2} + \frac{(\nabla\psi)^2}{2}\right]$$

② $\mathcal{H} = A\cdot B \cong \underset{\text{const.}}{B_z}\,\psi$

$$\Rightarrow\ H = \int d^3x\ B_z\psi, \quad \frac{dH}{dt} = 0, \text{ to } O(\eta)$$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\begin{cases} H \text{ conserved} \\ E_M \text{ dissipated} \end{cases}$

③ $K = \int d^3x\ \underline{V}\cdot\underline{B} = \int d^3x\ (\nabla\phi\cdot\underline{\nabla}\psi)$

also conserved, to dissipation.

Physics 218B

I.) Particle Orbits and Adiabatic Invariants

(1)

→ seek characterize particle orbits in complex magnetic and electric fields

→ proceed via:

1. guiding center concept
2. g.c. drifts — $\underline{E} \times \underline{B}$
   - polarization expansion
   - ∇B, curvature
3. adiabatic invariants
   - magnetic moment
   - general theory
4. particle motion in mirror, tokamak fields.

$\underline{B}_0 = B_0 \hat{z}$

i.) G.C. Concept

Crucial to orbit theory is notion of guiding center, i.e.

orbit = $\underset{\text{G.C.}}{\times}$ $\underline{\rho}$ Larmor precession at Ω, $\rho = \dfrac{v_\perp}{\Omega}$

$\Omega = \varepsilon B / mc$

slow fast

$$\therefore \quad \underset{\substack{\downarrow \\ \text{particle position}}}{\underline{x}} = \underset{\substack{\downarrow \\ \text{guiding center position}}}{\underline{X}_{gc}} + \underset{\substack{\downarrow \\ \text{Larmor precession}}}{\underline{\rho}}$$

i.e. $\underline{\rho} = \underline{\rho}_0 e^{i\Omega t}$

Utility: → for $t \gg \Omega^{-1}$, can average out fast cyclotron motion and keep track of g.c. dynamics alone
⇒ drift gyrokinetics ↔ kinetic equation for g.c.'s. – DKE/GKE

→ can simplify motion of particle in complex field (Electric, curved magnetic, etc) via g.c. dynamics.

ii.) G.C. Drifts

a.) First, take $\underline{B} = B_0 \hat{z}$, with $\underline{E}$

$$m \dot{\underline{v}} = q \underline{E} + q \frac{\underline{v} \times \underline{B}_0}{c}$$

$$\Rightarrow \quad m \dot{v}_{\parallel} = q E_{\parallel} \qquad (\cdot \ \underline{B}_0)$$

$$m \dot{\underline{v}}_{\perp} = q \underline{E}_{\perp} + q \frac{\underline{v}_{\perp} \times \underline{B}_0}{c} \qquad \left(\begin{array}{l} \text{for } \perp \\ \underline{v}_{\perp} = \underline{v} - \underline{v}_{\parallel} \end{array} \right)$$

$$\dot{\underline{v}}_{\perp} = \frac{q \underline{E}_{\perp}}{m} + \underline{v}_{\perp} \times \Omega \hat{z}$$

for g.c. motion $|\dot{V}_\perp|/|V_\perp| \ll \Omega$, so l.o.

l.o. $\quad 0 = \frac{q}{m}\underline{E}_\perp + \underline{V}_\perp \times \Omega\hat{z}$

$$\Rightarrow 0 = \hat{z} \times \frac{q}{m}\underline{E}_\perp + \hat{z} \times \underline{V}_\perp \times \Omega\hat{z}$$

$$\therefore \boxed{\begin{aligned} \underline{V}_\perp &= \frac{c}{B}\underline{E}_\perp \times \hat{z} \\ &= \frac{c}{B_0^2}\underline{E}_\perp \times \underline{B}_0 \end{aligned}} \Bigg\} \ \underline{E} \times \underline{B} \text{ drift.}$$

N.B. $\rightarrow \perp E, B$
$\rightarrow$ independent mass, charge
(i.e. all species $\underline{E}\times\underline{B}$ drift the same).

now, can extend, generalize to time varying field:

$$\dot{\underline{V}}_\perp = \frac{q}{m}\underline{E} + \underline{V} \times \Omega\hat{z}$$

$$\underline{V}_\perp = \underline{V}_\perp^{(0)} + \epsilon\, \underline{V}_\perp^{(1)} + \cdots$$

$\epsilon \sim O(\omega/\Omega)$
$\quad \hookrightarrow$ frequency of variation.

$$\underline{\dot{V}}_\perp^{(0)} + \epsilon\, \underline{\dot{V}}_\perp^{(1)} = \frac{q}{m}\underline{E} + \left(\underline{V}_\perp^{(0)} + \underline{V}_\perp^{(1)}\epsilon\right) \times \Omega \hat{z}$$

l.o. : $$0 = \frac{q}{m}\underline{E} + \underline{V}_\perp^{(0)} \times \Omega \hat{z}$$

$$\Rightarrow \quad \underline{V}_\perp^{(0)} = \frac{c}{B}\underline{E}_\perp \times \hat{z}$$

1st order : $$\underline{\dot{V}}_\perp^{(0)} = \underline{V}_\perp^{(1)} \times \Omega \hat{z}$$

polarization expansion

$$\Rightarrow \quad \underline{V}_\perp^{(1)} = \frac{1}{\Omega}\left(\hat{z} \times \underline{\dot{V}}_\perp^{(0)}\right)$$

$$\therefore \quad \boxed{\left.\begin{aligned} \underline{V}_\perp^{(1)} &= \frac{c}{B\Omega}\underline{\dot{E}}_\perp \\ &= \frac{c^2}{B^2 q} m \underline{\dot{E}}_\perp \end{aligned}\right\} \text{polarization drift}}$$

→ 2nd term in $O(\omega/\Omega)$ expansion (i.e. polarization expansion)

relate H-M

→ unlike $\underline{E} \times \underline{B}$ drift,

$$\underline{V}_\perp^{(1)} \sim \frac{m}{q}$$

→ larger for ions
→ direction ⟺ charge.

$$\rightarrow \quad \nabla_\perp \cdot V_\perp \cong \overset{0}{\cancel{\nabla\cdot\left(\frac{c}{B_0}\,E_\perp\times\hat{z}\right)}} + \nabla\cdot\left(\frac{c}{B\Omega_i}\,\dot{E}_\perp\right)$$

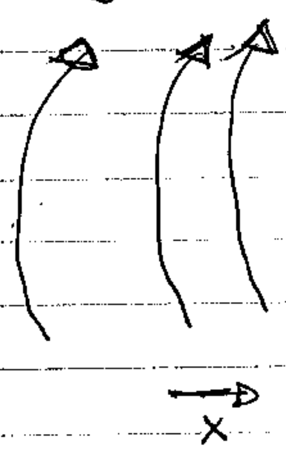
$$\underbrace{\nabla_\perp\cdot V_\perp = \frac{c}{B\Omega_i}\,\nabla\cdot\dot{\hat{E}}_\perp}_{H\text{-}14}$$ → compressibility of g.c. drift set by polarization

→ note that time averaged $V_\perp$ $\left(\langle\,\rangle = \int_0^T \frac{dt}{T}\right)$

$$\langle V_\perp\rangle = \frac{c}{B}\,E_\perp\times\hat{z}.$$

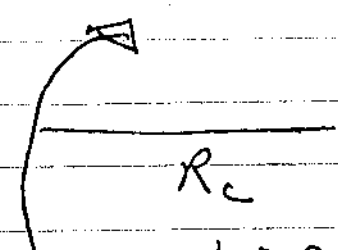
i.e. polarization et al. are $\frac{d}{dt}$

i) inhomogeneous magnetic field.



→ types of variation
- curvature
- increase of strength
- increase of strength along → mirroring

curvature:



↳ radius of curvature

then if particle/g.c. streams along field line, feels centrifugal force i.e. $\underline{F} = \frac{m v_\parallel^2}{R_c^2} \underline{R_c}$

then, if insert body force into Lorentz eqn. :

$$\frac{d\underline{v}}{dt} = \frac{\underline{F}}{m} + \Omega \, \underline{v} \times \hat{n} \qquad \hat{n} = \frac{\underline{B_0}}{|B_0|}$$

$$= \frac{m v_\parallel^2}{m R_c^2} \underline{R_c} + \Omega \, \underline{v} \times \hat{n}$$

(n.b. : concept of locally strong field)

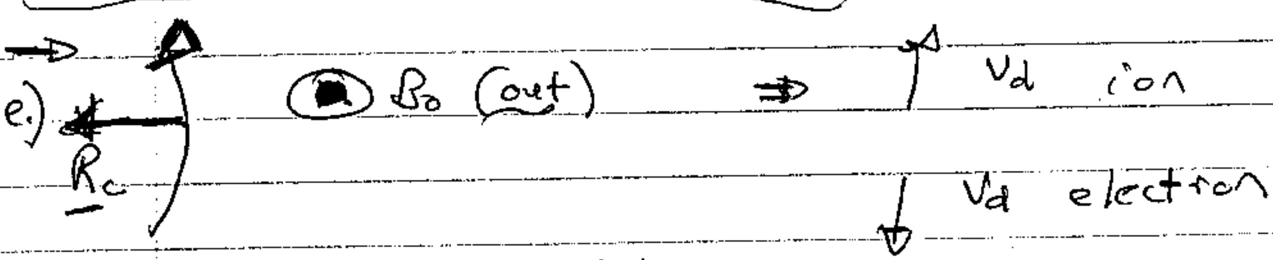
i.e. 'a la' $\underline{E} \times \underline{B}$ drift

$$\boxed{\underline{V}_d = \frac{v_\parallel^2}{\Omega} \frac{\underline{R_c} \times \underline{B}}{B R_c^2}}$$

$\underline{V}_d$: cross field drift

curvature drift

(i.e.)



opposite for diffnt species!

$\Rightarrow$ curvature drift induces charge separation!

$\rightarrow$ can re-write using:

$$\boxed{\underline{V_d} = \frac{V_\parallel^2}{\Omega} \frac{\hat{r}_c \times \hat{n}}{R_c}}$$

$\rightarrow$ curvature drift.

Similarly, can consider gradient in field strength

$$\dot{\underline{v}}_1 = \frac{q}{mc}\,\underline{v} \times \underline{B}_0(x)$$

expand and iterate ⇒

$$\underline{B}_0(x) \cong \underline{B}_0(0) + \underline{x}\cdot\frac{\partial}{\partial \underline{x}}\,\underline{B}_0\Big|_0 + \cdots$$

$$\underline{x} = \underline{x}_{gc} + \vec{\rho}$$

$$\cancel{\dot{\underline{x}}_{gc} + \dot{\vec{\rho}}} = \cancel{\frac{q}{mc}\,\underline{v}\times\underline{B}(0)} + \frac{qB}{mc}\,\underline{v}\times\left(\underline{x}\cdot\frac{1}{B}\frac{\partial \underline{B}_0}{\partial \underline{x}}\right)$$

for $\langle \dot{\underline{x}}_{gc} \rangle = \underline{V}_d$, avg ⇒

$$\underline{V}_d = \frac{qB}{mc}\left\langle \underline{v}\times\left(\frac{v_x}{\Omega}\,\frac{1}{B}\,\frac{\partial B_0}{\partial x}\right)\hat{z}\right\rangle_{\Omega^{-1}}$$

$$= \frac{1}{2}\,\frac{v_\perp^2}{\Omega}\,\frac{1}{B}\,\frac{\partial B}{\partial x}\,\hat{y}$$

more generally, for

Drift / Gyrokinetics

- now, Vlasov eqn ⇒ phase space density conserved along particle orbits

i.e.
$$\frac{\partial f}{\partial t} + \underbrace{\underline{v}}_{\frac{d\underline{x}}{dt}}\cdot\nabla f + \underbrace{\frac{q}{m}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)}_{\frac{d\underline{v}}{dt}}\cdot\nabla_{\underline{v}} f = 0$$

- but in real plasmas,

orbits complex ⇒ drifts, etc.

∴ (f).

- convenient to work with phase space density of guiding centers ⇒ drift kinetics

⇒ useful for phenomena with $\omega \ll \Omega$, $k\rho \ll 1$.

∴ can, in a sense 'write down' DKE for particles in $B_0 = B_0\hat{z}$ and electrostatic fluctns using drift eqns as characteristics

i.e. $\frac{dz}{dt} = v_z$ $\qquad \frac{d\underline{x}_\perp}{dt} = \frac{c}{B}\,\underline{E}\times\hat{z}$

$\frac{dv_z}{dt} = \frac{q}{m}E_z$

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial z}\left(\frac{dz}{dt}f\right) + \frac{\partial}{\partial \underline{x}_\perp}\cdot\left(\left[\frac{c}{B_0}\underline{E}\times\hat{z}\right]f\right)$$

$$+ \frac{\partial}{\partial v_z}\left(\frac{q}{m}E_z f\right) = 0$$

curved B ?

$\underline{\nabla}_\perp \cdot \underline{V}_{E\times B} = 0 \quad \Rightarrow$

$$\boxed{\frac{\partial f}{\partial t} + v_z\frac{\partial f}{\partial z} + \frac{c}{B}\,\underline{E}\times\hat{z}\cdot\underline{\nabla} f + \frac{q}{m}E_z\frac{\partial f}{\partial v_z} = 0}$$

- drift kinetic equation

- useful with additions in describing electron ($k_\perp\rho_e \to 0$) behavior in MHD, low frequency modes.

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Linearized gyro-kinetics

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Abstract—Finite gyroradius effects are retained in a far simpler manner than previous treatments by transforming to the guiding center variables and gyro-averaging *before* introducing magnetic coordinates.

MANY INSTABILITIES of interest in present and future magnetic confinement devices depend sensitively on finite gyroradius effects. As a result, it is important to develop techniques which *simply* retain finite gyroradius effects in complicated magnetic fields. When the time variation of the waves is slow compared to the gyrofrequency and the gyroradius small compared to unperturbed scale lengths, then gyro-kinetic techniques may be employed to retain gyro-effects for arbitrary values of the gyroradius over the perpendicular wavelength. Unlike drift kinetic descriptions, gyro-kinetic techniques retain finite gyroradius effects to lowest order in the equation for the perturbed or linearized distribution function.

The original gyro-kinetic work of RUTHERFORD and FRIEMAN (1968) and TAYLOR and HASTIE (1968) considers general geometries but employs a WKB or eikonal assumption for the spatial variation of the electrostatic potential Φ. Later work (JAMIN, 1971; CONNOR and HASTIE, 1975; and NEWBERGER, 1976) also employs the eikonal ansatz, with CONNOR and HASTIE being the first to employ a form for Φ satisfying both poloidal and toroidal periodicity constraints for axisymmetric geometries with finite magnetic shear. Later work by CATTO and TSANG (1977) attempts to remove the WKB assumption by employing a concentric magnetic surface model (KADOMSTEV and POGUTSE, 1969 and 1967).

In all of the preceding work magnetic coordinates were introduced prior to making the transformation to the guiding center variables. It is the transformation from the particle variables to the guiding center variables which permits finite gyroradius effects to be retained in lowest order. The present treatment avoids the substantial mathematical complications inherent in these prior treatments by introducing the transformation to the guiding center variables and performing the guiding center gyrophase average *before* specifying the magnetic coordinates to be employed. In this way the unperturbed, gyro-averaged Vlasov operator which retains finite gyro-effects is obtained in the most convenient manner for arbitrary unperturbed magnetic fields.

After the transformation has been performed, the magnetic coordinates can be introduced with relative simplicity in order to obtain the appropriate gyro-averaged Φ. The magnetic variables are only necessary to evaluate the gyro-average of the inhomogeneous term in the linearized Vlasov equation. In order to evaluate the inhomogeneous term a mode structure for Φ is required. To illustrate the technique to completion a Tokamak geometry is considered. A WKB or eikonal ansatz for the poloidal mode structure is avoided by employing a poloidal angle variable so that the poloidal variation can be strictly Fourier decomposed (TANG *et al.*, 1977).

A gyro-kinetic description is obtained by employing the guiding center variables $\mathbf{R}$, $\mathbf{E}$, μ, and ϕ where the guiding center variables are related to the original particle variables $\mathbf{r}$ and $\mathbf{v}$ via

$$\mathbf{R} = \mathbf{r} + \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}},$$
$$\mathbf{v} = v_\parallel\hat{\mathbf{n}} + (\hat{\boldsymbol{\psi}}\cos\phi + \hat{\mathbf{e}}\sin\phi) = v_\parallel\hat{\mathbf{n}} + v_\perp\hat{\mathbf{v}}_\perp, \tag{1}$$

with $E = v^2/2$, $\mu = v_\perp{}^2/2B$, $v_\parallel{}^2 = 2(E-\mu B)$, $v_\perp{}^2 = [(\mathbf{I}-\hat{\mathbf{n}}\hat{\mathbf{n}})\cdot\mathbf{v}]^2$, $B = |\mathbf{B}|$, $\hat{\mathbf{n}} = \mathbf{B}/B$, $\Omega = ZeB/Mc$, $v = |\mathbf{v}|$, and the unit vectors $\hat{\boldsymbol{\psi}}$, $\hat{\mathbf{e}}$, and $\hat{\mathbf{n}}$ forming an orthogonal system in which $\hat{\mathbf{e}} = \hat{\mathbf{n}}\times\hat{\boldsymbol{\psi}}$. The quantities Z and M are the species charge number and mass; e and c are the magnitude of the charge on an electron and the speed of light. Unlike previous treatments, magnetic coordinates have not been explicitly introduced, thereby drastically simplifying the analysis that follows.

Changing to the guiding center variables, $\partial/\partial t \rightarrow \partial/\partial t$ while

$$\nabla \rightarrow \nabla_R - [\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot\nabla_R + \nabla\phi\frac{\partial}{\partial\phi} + \nabla\mu\frac{\partial}{\partial\mu},$$
$$\nabla_v \rightarrow \nabla_V + \Omega^{-1}\mathbf{I}\times\hat{\mathbf{n}}\cdot\nabla_R, \tag{2}$$

with $\nabla_V = \mathbf{v}\partial/\partial E + \mathbf{v}_\perp B^{-1}\partial/\partial\mu + \hat{\boldsymbol{\phi}} v_\perp^{-1}\partial/\partial\phi$, $\hat{\boldsymbol{\phi}} = \hat{\mathbf{n}} \times \hat{\mathbf{v}}_\perp$, $\mathbf{I}$ the unit dyadic, and

$$\nabla\phi = (v_\parallel/v_\perp)[-(\nabla\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}\cos\phi + (\nabla\hat{\mathbf{n}})\cdot\hat{\boldsymbol{\psi}}\sin\phi] + (\nabla\hat{\mathbf{e}})\cdot\hat{\boldsymbol{\psi}} \tag{3}$$

$$\nabla\mu = -(\mu/B)\nabla B - (v_\perp v_\parallel/B)[(\nabla\hat{\mathbf{n}})\cdot\hat{\boldsymbol{\psi}}\cos\phi + (\nabla\hat{\mathbf{n}})\cdot\hat{\mathbf{e}}\sin\phi]. \tag{4}$$

Equations (3) and (4) are obtained by operating on $\hat{\boldsymbol{\psi}}\cdot\mathbf{v} = v_\perp\cos\phi$ and $\hat{\mathbf{e}}\cdot\mathbf{v} = v_\perp\sin\phi$ with ∇ and ∇_v and forming the appropriate combinations.

The unperturbed Vlasov operator in the guiding center variables is obtained from the change of variables (1) by noting that $(Ze/Mc)\mathbf{v}\times\mathbf{B}\cdot\nabla_v = -\Omega\partial/\partial\phi$ and $(\Omega\mathbf{v}\times\hat{\mathbf{n}})\cdot(\Omega^{-1}\mathbf{I}\times n\cdot\nabla_R) = -\mathbf{v}_\perp\cdot\nabla_R$ so that

$$\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + (Ze/Mc)\mathbf{v}\times\mathbf{B}\cdot\nabla_v \rightarrow -\Omega\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t} + v_\parallel\hat{\mathbf{n}}\cdot\nabla_R + \mathbf{v}\cdot\left[\nabla\phi\frac{\partial}{\partial\phi} + \nabla\mu\frac{\partial}{\partial\mu} - \nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}\cdot\nabla_R\right]. \tag{5}$$

For time variations slow compared to the gyromotion the dominant term in (5) is $-\Omega\partial/\partial\phi$ so that to lowest order the variable operated on by (5), usually a portion of distribution function, must be independent of ϕ. When the quantity being operated on is independent of the guiding center gyrophase, the unperturbed Vlasov operator may be gyro-averaged by employing $\mathbf{v}\cdot\nabla\phi\,\partial/\partial\phi \rightarrow 0$,

$$(2\pi)^{-1}\oint d\phi\,\mathbf{v}\mathbf{v} = (v_\perp^2/2)(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}) + v_\parallel^2\hat{\mathbf{n}}\hat{\mathbf{n}},$$

$$\mathbf{v}_d = -(2\pi)^{-1}\oint d\phi\{\mathbf{v}\cdot[\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}})\} = \mathbf{n}\times[(\mathbf{v}_\perp^2/2\Omega)\nabla\ln B + (v_\parallel^2/\Omega)\hat{\mathbf{n}}\cdot\nabla\hat{\mathbf{n}}] \tag{6}$$

$$u_\parallel = -(2\pi)^{-1}\oint d\phi\{\mathbf{v}\cdot[\nabla(\Omega^{-1}\hat{\mathbf{n}})\times\mathbf{v}]\cdot\mathbf{n}\} = -(v_\perp^2/2\Omega)\hat{\mathbf{n}}\cdot\nabla\times\hat{\mathbf{n}}, \tag{7}$$

and $(2\pi)^{-1}\oint d\phi\,\mathbf{v}\cdot\nabla\mu \approx 0$ to obtain

$$(2\pi)^{-1}\oint d\phi\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \left(\frac{Ze}{Mc}\right)\mathbf{v}\times\mathbf{B}\cdot\nabla_v\right] \rightarrow \frac{\partial}{\partial t} + (v_\parallel\hat{\mathbf{n}} + \mathbf{v}_d)\cdot\nabla_R. \tag{8}$$

In writing (8), the parallel velocity correction (Hazeltine, 1973) $u_\parallel$ must be neglected because it is the same order as the gyroradius over scale length corrections to $\oint d\phi\,\mathbf{v}\cdot\nabla\mu$. The curvature and ∇B drifts, $\mathbf{v}_d$, are retained because perpendicular wavelengths are assumed small compared to parallel wavelengths which are of the order of unperturbed scale lengths. It should be noted that the variable operated on by (8) is a function of $\mathbf{R}$, E, and μ but not ϕ. Once this variable has been solved for in terms of the guiding center location $\mathbf{R}$ one must revert to the original particle variable $\mathbf{r} = \mathbf{R} - \Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}$, thereby introducing particle gyrophase dependence via the $\mathbf{v} = v_\parallel\hat{\mathbf{n}} + v_\perp(\hat{\boldsymbol{\psi}}\cos\phi + \hat{\mathbf{e}}\sin\phi)$. Because only the lowest order ϕ dependence is needed, distinctions between particle and guiding center gyrophase, energy, and magnetic moment are not required.

It should be noted that the gyro-kinetic technique outlined in the preceding paragraphs has not required that the magnetic coordinates be identified before carrying out the change of variables and gyro-average. The magnetic coordinates need only be introduced when it becomes necessary to evaluate the gyro-average of the inhomogeneous term in the linearized Vlasov equation. The technique is illustrated in the following paragraphs for a Tokamak geometry.

The unperturbed distribution function F_0 is taken to be a function of v^2 and the canonical angular momentum $(-Ze/c)\psi_0$, with $\psi_0 = \psi - (cMR/Ze)\hat{\boldsymbol{\zeta}}\cdot\mathbf{v}$, such that

$$F_0 = F_0(\psi_0, v^2) = N(\psi_0)[M/2\pi T(\psi_0)]^{3/2}\exp[-Mv^2/2T(\psi_0)]$$
$$\approx F_M\left\{1 - \hat{\boldsymbol{\zeta}}\cdot\mathbf{v}\frac{McR}{ZeN}\frac{\partial N}{\partial\psi}\left[1 + \eta\left(\frac{Mv^2}{2T} - \frac{3}{2}\right)\right]\right\}, \tag{9}$$

with $\eta = d\ln T/d\ln N$, $F_M \equiv F_0(\psi, v^2)$, and $N(\psi)$ and $T(\psi)$ the density and temperature as a function of poloidal flux ψ. The second form of (9) follows from a Taylor expansion of F_0 about $\psi_0 = \psi$, and $\boldsymbol{\zeta}$ and R are the toroidal angle variable and the distance from the axis of symmetry to the point of interest ($|\nabla\zeta| = 1/R$).

Defining the non-adiabatic portion of the perturbed distribution function g in terms of the perturbed distribution function f via $g = f + (ZeF_0/T)\Phi$, the linearized Vlasov equation may be written as

$$\frac{\partial g}{\partial t} + \mathbf{v}\cdot\nabla g + \frac{Ze}{Mc}\mathbf{v}\times\mathbf{B}\cdot\nabla_v g = \frac{Ze}{T}F_0\frac{\partial\Phi}{\partial t} - c\frac{\partial F_0}{\partial\psi_0}\frac{\partial\Phi}{\partial\zeta}. \tag{10}$$

Employing the gyro-kinetic variables of (1) and considering time variations slow compared to a gyro-period, (10) yields $\partial g/\partial\phi = 0$ to lowest order. As a result, applying the results of the gyro-kinetic section to the next order equation for g results in the gyro-averaged equation for g.

$$\frac{\partial g}{\partial t}+(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\nabla_R g=\Omega\oint\frac{\mathrm{d}\phi}{2\pi\Omega}\left[\frac{ZeF_0}{T}\frac{\partial\Phi}{\partial t}(\mathbf{r},t)_{-c}\frac{\partial F_0}{\partial\psi_0}\frac{\partial\Phi}{\partial\zeta}(\mathbf{r},t)\right]. \tag{11}$$

In order to simplify the right-hand side of (11), the magnetic coordinates ψ, Θ, ζ are introduced, where $-\pi<\Theta\leq\pi$ is a poloidal angle variable. The potential $\Phi(\mathbf{r},t)$ is assumed to be of the form

$$\Phi(\mathbf{r},t)=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L \mathrm{d}\omega\,\Phi_{lm}(\kappa,\omega)\exp\left(-i\omega t+i\kappa\psi+im\Theta-il\zeta\right), \tag{12}$$

where L is the Landau contour (above all singularities in the complex plane). Away from the magnetic axis a change of variables from $\mathbf{R}$ to ψ', Θ', ζ' where $\psi'=\psi+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\psi$, $\Theta'=\Theta+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\Theta$, and $\zeta'=\zeta+\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\zeta$ (Jamin, 1971 and Newberger, 1976) gives $\exp i(\kappa\psi+m\Theta-l\zeta)=\exp[i(\kappa\psi'+m\Theta'-l\zeta')-i\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\mathbf{k}]$ with $\mathbf{k}\equiv\kappa\nabla\psi+m\nabla\Theta-l\nabla\zeta$ and $\psi_0=\psi'+(cMR/Ze)\hat{\zeta}\cdot\mathbf{v}-\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\nabla\psi$. Expanding $F_0(\psi_0,v^2)$ and $\partial F_0(\psi_0,v^2)/\partial\psi_0$ about $\psi_0=\psi'$, neglecting gyroradius over unperturbed scale length corrections and employing $(2\pi)^{-1}\oint\mathrm{d}\phi\exp(-i\Omega^{-1}\mathbf{k}\cdot\mathbf{v}\times\mathbf{n})=J_0(k_\perp v_\perp/\Omega)$ so that

$$\bar{\Phi}\equiv(2\pi)^{-1}\oint\mathrm{d}\phi\,\Phi(\mathbf{r},t)$$

$$=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega\,\Phi_{lm}J_0(k_\perp v_\perp/\Omega)\exp\left(-i\omega t+i\kappa\psi'+im\Theta'-il\zeta'\right) \tag{13}$$

with $k_\perp=|\hat{\mathbf{n}}\times(\mathbf{k}\times\hat{\mathbf{n}})|$, then (11) becomes

$$\frac{\partial g}{\partial t}+(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)\cdot\nabla_R g=\frac{ZeF_M}{T}\left\{\frac{\partial\bar{\Phi}}{\partial t}-\frac{cT}{ZeN}\frac{\partial N}{\partial\psi'}\left[1+\eta\left(\frac{Mv^2}{2T}-\frac{3}{2}\right)\right]\frac{\partial\bar{\Phi}}{\partial\zeta'}\right\}, \tag{14}$$

where all unperturbed quantities are functions of the primed variables.

In general (14) must be solved by integrating along the unperturbed guiding center trajectories. In order to further illustrate the treatment of the gyromotion effects, $v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d$ will be assumed to be independent of ψ' and Θ'. This approximation is the one usually employed in evaluating the ion response for the trapped electron and collisionless drift instabilities. For $v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d$ a constant and neglecting the ψ dependence of the right side of (14), g may be taken to be

$$g=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega\,g'_{lm}(\kappa,\omega)\exp\left(-i\omega t+i\kappa\psi'+im\Theta'-il\zeta'\right). \tag{15}$$

Employing (13) and (15), equation (14) may be transformed to obtain

$$g_{lm}'(\kappa,\omega)=\frac{(ZeF_M/T)(\omega-\omega_*^T)J_0(k_\perp v_\perp/\Omega)}{\omega-\mathbf{k}\cdot(v_\parallel\hat{\mathbf{n}}+\mathbf{v}_d)}\Phi_{lm}(\kappa,\omega) \tag{16}$$

where $\omega_*^T(\psi)=(lcT/ZeN)(\partial N/\partial\psi)\{1+\eta[(Mv^2/2T)-(3/2)]\}$. Reverting to the unprimed variables ψ, Θ, ζ by using (15),

$$g=(2\pi)^{-2}\sum_{l,m}\int_{-\infty}^{\infty}\mathrm{d}\kappa\int_L\mathrm{d}\omega\,g_{lm}(\kappa,\omega)\exp\left(-i\omega t+i\kappa\psi+im\Theta-il\zeta\right), \tag{17}$$

with $g_{lm}(\kappa,\omega)=g_{lm}'(\kappa,\omega)\exp(i\Omega^{-1}\mathbf{v}\times\hat{\mathbf{n}}\cdot\mathbf{k})$. The g from (17) is in terms of particle variables while that from (15) is in terms of the guiding center variables. The gyrophase dependence $\exp(i\Omega^{-1}\mathbf{k}\cdot\mathbf{v}\times\hat{\mathbf{n}})$ in $g_{lm}(\kappa,\omega)$ enters as it does when a trajectory integral over the full gyromotion is performed. Using $f_{lm}=-(ZeF_M/T)\Phi_{lm}+g_{lm}$ to form the perturbed density $n_{lm}=\int\mathrm{d}^3v f_{lm}$ and employing ϕ, $v_\perp$, $v_\parallel$ variables for the velocity integrations results in

$$n_{lm}=-\frac{ZeN}{T}\Phi_{lm}\left[1+\frac{\Gamma_0}{|k_\parallel|v_T}\left(\left\{\omega-\omega_*\left[1-\eta\left(\frac{1}{2}+b-\frac{b\Gamma_1}{\Gamma_0}-\xi^2\right)\right]\right\}Z(\xi)-\eta\omega_*\xi\right)\right], \tag{18}$$

with $\omega_*=(lcT/ZeN)(\partial N/\partial\psi)$, $v_T{}^2=2T/M$, $\Gamma_\nu=I_\nu(b)\exp(-b)$, $b=k_\perp{}^2T/M\Omega^2=k_\perp{}^2v^2/2\Omega^2$, $\xi=(\omega-\mathbf{k}\cdot\mathbf{v}_d)/|k_\parallel|v_T$, and $Z(\xi)$ the usual plasma dispersion function.

When the ψ variation of Φ is slow compared to a gyroradius then $k_\perp{}^2>\kappa^2|\nabla\psi|^2$ may be employed to expand Γ_ν about the poloidal wave vector squared, $k_p{}^2 = k_\perp{}^2 - \kappa^2 |\nabla\psi|^2$. If only terms to order κ^2 are retained then the inverse transform form κ space back to ψ will recover the second derivatives of $\Phi_{lm}(\kappa, \omega)$ with respect to ω because $\kappa^2 \to -\partial^2/\partial\psi^2$.

In summary, the linearized, unperturbed, gyro-averaged Vlasov operator containing finite gyroradius effects is obtained by a substantially simpler method than previous work, the technique transforms from the particle variables to the guiding center variables and gyro-averages before (rather than after) explicitly introducing the magnetic coordinates.

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II.)
→ **Gyrokinetics** : $\left\{\begin{array}{l}\text{Key ordering for magnetized} \\ \text{plasma with } \rho/L_\perp \ll 1.\end{array}\right.$

- seek employ drifts, with gyro-radius finite, to reduce description of plasma dynamics from Vlasov → Gyrokinetic.

i.e. DKE: (GKE with $k_\perp \rho \to 0$)

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B_0} \nabla\phi \times \hat{z} \cdot \nabla f + \frac{q}{m} E_z \frac{\partial f}{\partial v_z} = C(f)$$

1D motion + $E \times B$ drift replaces 6D phase space.

- central idea is guiding center transformation:

$$\underline{x} = \underline{X}_{g.c.} - \frac{\underline{v} \times \hat{n}}{\Omega_{\text{cyclotron}}}$$

$$v_\perp = \hat{v}\cos\phi + \hat{e}\sin\phi$$

$$\hat{e} = \hat{n} \times \hat{v}$$

$$\underline{v} = \underbrace{v_\parallel \hat{n}}_{\text{along field}} + \underbrace{v_\perp \hat{v}_\perp}_{\perp \text{ velocity}}$$

$\hat{n} = \underline{B}/|B|$

where: $E = v^2/2$ → energy $\qquad \Omega = qB/mc$

$$\mu = v_\perp^2/2B \qquad v_\perp^2 = \left[(\underline{\underline{I}} - \hat{n}\hat{n}) \cdot \underline{v}\right]^2$$

$$\underbrace{v_\parallel^2 = 2(E - \mu B)}_{\text{velocity variables}} \qquad = \left[\underline{v} - \hat{n}(\underline{v} \cdot \hat{n})\right]^2$$

for linearized theory:

→ simplifying LHS: $\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v$

Now, changing to g.c. variables:

$$\underline{\nabla} \to \underline{\nabla}_R - \left[\nabla(\Omega^{-1}\hat{n})\times\underline{v}\right]\cdot\underline{\nabla}_R + \underline{\nabla}\phi\frac{\partial}{\partial\phi} + \underline{\nabla}\mu\cdot\frac{\partial}{\partial\mu}$$

$$\underline{\nabla}_v \to \underline{\nabla}_V + \frac{\underline{\underline{I}}\times\hat{n}}{\Omega}\cdot\underline{\nabla}_R$$

so

$$\underline{\nabla}_v = \underline{v}\frac{\partial}{\partial E} + \frac{\hat{\underline{v}}_\perp}{B}\frac{\partial}{\partial\mu} + \frac{\vec{\phi}}{v_\perp}\frac{\partial}{\partial\phi} \qquad \hat{\phi} = \hat{n}\times\hat{v}_\perp$$

$$\underline{\nabla}\mu = -(\mu/B)\underline{\nabla}B - \left(\frac{v_\perp v_\parallel}{B}\right)\left[(\nabla\hat{n})\cdot\vec{\psi}\cos\phi + (\nabla\hat{n})\cdot\underline{e}\sin\phi\right]$$

$$\underline{\nabla}\phi = \frac{v_\parallel}{v_\perp}\left[-(\nabla\hat{n})\cdot\hat{e}\cos\phi + (\nabla\hat{n})\cdot\vec{\psi}\sin\phi\right] + \underline{\nabla}\vec{e}\cdot\vec{\psi}$$

so

$$\frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v = -\Omega\frac{\partial}{\partial\phi} \quad \to \text{reduces to cyclotron motion}$$

and $\left(\Omega\underline{v}\times\hat{n}\right)\cdot\left(\frac{\underline{\underline{I}}\times\hat{n}}{\Omega}\cdot\underline{\nabla}_R\right) = -\underline{v}_\perp\cdot\underline{\nabla}_R$

↳ cancels ⊥ motion (fast term)

⇒

$$\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \frac{q}{mc}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v = -\Omega\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t} + v_\parallel\hat{n}\cdot\underline{\nabla}_R$$

↑ single fast term

$$+\,\underline{v}\cdot\left[\underline{\nabla}\phi\frac{\partial}{\partial\phi} + \underline{\nabla}\mu\frac{\partial}{\partial\mu} - \nabla(\Omega^{-1}\hat{n})\times\underline{v}\cdot\underline{\nabla}_R\right]$$

Now, can simplify for low frequency dynamics via

$$L = \frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla} + \frac{q}{m}\underline{v}\times\underline{B}\cdot\underline{\nabla}_v = -\Omega\frac{\partial}{\partial\phi} + \underbrace{L_s}_{\text{slow piece}}$$

$L\,f = RHS$, in Vlasov Eqn.

$\Rightarrow$

$$-\Omega\frac{\partial}{\partial\phi}f + L_s f = RHS$$

l.o. $\quad -\Omega\frac{\partial f^{(0)}}{\partial\phi} = 0 \quad \Rightarrow$ f independent ϕ

1st. ord.

$$-\Omega\frac{\partial f^{(1)}}{\partial\phi} + L_s f_0 = RHS$$

f_0 indep. ϕ

$\oint\frac{d\phi}{2\pi} \quad \Rightarrow \quad \langle L_s\rangle f_0 = \langle RHS\rangle$

where: $\langle L_s\rangle = \int\frac{d\phi}{2\pi}L_s$

$$\langle L_s\rangle f_0 = \langle RHS\rangle$$

"is" gyrokinetic equation.

Thus, remains to compute $\langle L_s\rangle f_0$, $\langle RHS\rangle$!

$$L_s = -\Omega \frac{\partial}{\partial\phi} + \frac{\partial}{\partial t} + v_{\parallel}\hat{n}\cdot\nabla_R$$
$$+ \underline{v}\cdot\left[\nabla\phi\frac{\partial}{\partial\phi} + \nabla\mu\frac{\partial}{\partial\mu} - \nabla(\Omega^{-1}\hat{n})\times\underline{v}\cdot\nabla_R\right],$$

so

$$\langle L_s\rangle = \frac{\partial}{\partial t}\langle\ \rangle + \langle v_{\parallel}\hat{n}\cdot\nabla_R\rangle$$
$$+\left\langle \underline{v}\cdot\left[\nabla\phi\frac{\partial}{\partial\phi} + \nabla\mu\frac{\partial}{\partial\mu} - \underbrace{\nabla(\Omega^{-1}\hat{n})\times\underline{v}\cdot\nabla_R}_{\text{drift piece}}\right]\right\rangle$$

i.e. $\nabla(1/\Omega)$

Now, can note:

$$\frac{1}{2\pi}\oint d\phi = \langle\ \rangle\ /$$

and

$$\langle \underline{v}\,\underline{v}\rangle = \frac{v_\perp^2}{2}(\underline{\underline{I}} - \hat{n}\hat{n}) + v_{\parallel}^2\hat{n}\hat{n}$$

so

$$-\langle \underline{v}\cdot[\nabla(\Omega^{-1}\hat{n})\times\underline{v}]\cdot(\underline{\underline{I}}-\hat{n}\hat{n})\rangle$$
$$= \hat{n}\times\left[\underbrace{(v_\perp^2/2\Omega)\nabla\ln B}_{\underline{v}_{\nabla B}} + \underbrace{(v_{\parallel}^2/\Omega)\hat{n}\cdot\nabla\hat{n}}_{\underline{v}_{curv}}\right].$$

$$\equiv \langle \underline{v}_d\cdot\nabla_\perp\rangle\ /$$

N.B. neglected $\langle v_d\cdot\nabla_{\parallel}\rangle$, as $k_\perp \gg k_{\parallel}$ assumed.

similarly;

$$\left. \begin{array}{l} \left\langle \underline{v} \cdot \nabla\phi \dfrac{\partial}{\partial\phi} \right\rangle = 0 \\ \left\langle \underline{v} \cdot \nabla \mu \right\rangle = 0 \end{array} \right\} \Rightarrow \text{symmetry}$$

Thus

$$\boxed{\langle L_0 \rangle = \frac{\partial}{\partial t} + v_{\parallel} \hat{n} \cdot \underline{\nabla}_R + \underline{v}_d \cdot \underline{\nabla}_R}$$

Now, what of RHS? ⇒

$$\therefore 2 \text{ issues} \left\langle \begin{array}{l} f \text{ vs } g \\ \text{density gradient} \end{array} \right.$$

① f vs. g ?

Can always write:

$$f = \underbrace{-\frac{q\phi}{T}\langle f \rangle}_{\substack{\text{Boltzmann} \\ \text{(adiabatic)}}} + \underbrace{g}_{\text{(non-adiabatic)}}$$

meaning: "adiabatic" ⇒ $k_{\parallel} v_{\parallel} \to \infty$
$k_{\parallel} v_{\parallel} > \omega$, etc.

piece.

i.e. – corresponds to particles zipping along field lines

$f = \frac{e\phi}{T} f_0 + g$

- corresponds to all = adiabatic + everything else

- adiabatic particles should not drift, etc. $\Rightarrow$ zip along field lines, dominantly

- utility: Current driven ion-acoustic

$v_{Ti} < \frac{\omega}{k_\parallel} < v_{Te}$ — prototype of one species adiabatic, one hydro.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{|e|}{m_e}E\frac{\partial f}{\partial v} = 0$$

$$f = \langle f \rangle + \hat{f}$$

$$\frac{\partial \hat{f}}{\partial t} + v\frac{\partial \hat{f}}{\partial x} = \frac{|e|}{m_e}\hat{E}\frac{\partial \langle f \rangle}{\partial v}$$

$$\langle f \rangle = \frac{n}{\sqrt{2\pi}}\exp\left[-(v-u_0)^2/2v_{th}^2\right]$$

$$\Rightarrow \frac{\partial \hat{f}}{\partial t} + v\frac{\partial \hat{f}}{\partial x} = \frac{|e|}{\cancel{m_e}}\frac{(v-u_0)}{T_e/\cancel{m_e}}\left(-\frac{\partial \phi}{\partial x}\right)\langle f \rangle$$

Now $\hat{f} = \frac{|e|\phi}{T}\langle f \rangle + \hat{g}$

$$\frac{\partial \hat{g}}{\partial t} + v\frac{\partial \hat{g}}{\partial x} = -v\frac{|e|}{T}\frac{\partial \hat{\phi}}{\partial x}\langle f \rangle - \frac{|e|}{T}\frac{\partial \phi}{\partial t}\langle f \rangle$$

$$+ \frac{|e|}{T}(v-u_0)\left(-\frac{\partial \phi}{\partial x}\right)\langle f \rangle$$

$$= -\frac{|e|}{T}\frac{\partial \hat{\phi}}{\partial t}\langle f \rangle + \frac{|e|}{T}u_0\frac{\partial \phi}{\partial x}\langle f \rangle$$

$$\sim (\omega - k u_0)$$

∴ of clear utility to isolate non-adiabatic piece, as

RHS $\sim \frac{\omega}{k} - u_0 \Rightarrow$ flips sign!

② why not drift?

Consider drift-kinetic-equation:

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} - \frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} f - \frac{|e|}{m} E_z \frac{\partial f}{\partial v_z} = 0$$

$k_\perp \gg k_\parallel \Rightarrow$ ($E_\parallel$ nonlinearity small)

$$\frac{\partial \tilde{f}}{\partial t} + v_z \frac{\partial \tilde{f}}{\partial z} - \frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} \tilde{f}$$

$$= \frac{c}{B} \underline{\nabla} \tilde{\phi} \times \hat{z} \cdot \underline{\nabla} \langle f \rangle + \frac{|e|}{m} \tilde{E}_z \frac{\partial \langle f \rangle}{\partial v_z}$$

$$E_z = -\frac{\partial \phi}{\partial z} \qquad f = \frac{|e| \tilde{\phi}}{T} \langle f_0 \rangle + g$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\left(\frac{|e|\tilde{\phi}}{T}\langle f_0 \rangle + \tilde{g}\right) + v_z \frac{\partial}{\partial z}\left(\frac{|e|\tilde{\phi}}{T}\langle f_0 \rangle + \tilde{g}\right)$$

$$- \frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla}\left(\cancel{\frac{|e|\tilde{\phi}}{T}\langle f_0 \rangle} + \tilde{g}\right) = \frac{c}{B} \underline{\nabla} \tilde{\phi} \times \hat{z} \cdot \underline{\nabla} \langle f \rangle + \frac{|e|}{m} \tilde{E}_z \frac{\partial \langle f \rangle}{\partial v_z}$$

NL ($E \times B$ drift) annihilates adiabatic part only non-adiabatic part advanced.

$\Rightarrow$

$$\frac{\partial \vec{g}}{\partial t} + v_z \frac{\partial}{\partial z} \vec{g} - \frac{c}{B_0} \nabla\phi \times \hat{z} \cdot \nabla \vec{g}$$

$$= \frac{-|e|}{T} \langle f_0 \rangle \frac{\partial \vec{\phi}}{\partial t} + \frac{c}{B} \nabla\phi \times \hat{z} \cdot \nabla \langle f \rangle$$

$\therefore$ bottom line, in electrostatics is:

$+ \rightarrow$ ions
$- \rightarrow$ electrons

$$RHS = \frac{\pm|e|}{T_0} \ddot{f_0} \frac{\partial \langle \phi \rangle}{\partial t} + \frac{c}{B} \nabla\langle\phi\rangle \times \hat{z} \cdot \nabla \langle f \rangle$$

$\rightarrow$

meaning $\langle \; \rangle$?

$$\phi = \sum_{\underline{k}} \phi_{\underline{k}} \, e^{i\underline{k}\cdot\underline{x} - i\omega t}$$

$$\underline{x} = \underline{R} - \frac{\underline{v} \times \hat{n}}{\Omega}$$

$\Rightarrow$

$$\underline{k}\cdot\underline{x} = \underline{k}\cdot\underline{R} - \underline{k}\cdot\frac{\underline{v}\times\vec{n}}{\Omega}$$

$$\psi = \measuredangle k$$

$$= k\cdot R + k\rho \sin(\phi - \psi)$$

$$\langle \phi_k \rangle = \frac{1}{2\pi} \oint d\phi \, \phi_k \, e^{i\underline{k}\cdot\underline{R}} \, e^{ik\rho \sin(\phi - \psi)}$$

but $\oint \frac{d\phi}{2\pi} e^{ik\rho \sin(\phi - \psi)} \equiv J_0(k_\perp \rho)$

or $\hat{n}_i = \hat{n}_{ac} - i k^2 \rho_i^2 \frac{e\phi}{T} = \hat{n}_{ac} + k^2 \rho^2 \frac{e\phi}{T}$

$\hat{n}_e = \frac{e\phi}{T}$

so in fourier components, can write gyrokinetic equation as:

$$-i\left(\omega - \underline{k}_\perp \cdot \underline{V}_d - k_\parallel v_\parallel\right) \hat{g}_{\underline{n}}$$

gyro- [illegible] avg. factor

comment → DKE → Johnel

$$= \mp \frac{i|e|}{T_{i,e}} \left(\omega - \omega_{*i,e}\right) f_0 \, J_0(k_\perp \rho) \, \phi_{\underline{n}}$$

$$\omega_* = \frac{1}{\langle f \rangle} \frac{Tc}{B_0 q} k_\theta \frac{\partial \langle f \rangle}{\partial r}$$

$$= + k_\theta \left(\frac{(T/m)\, mc}{q B_0}\right) = k_\theta \rho V \frac{1}{\langle f_0 \rangle} \frac{\partial \langle f_0 \rangle}{\partial r}$$

$$\frac{\partial f_0}{\partial r} = \frac{\partial}{\partial r} \frac{n\, e^{-(v - v_{\parallel i})^2 / v_{th}^2}}{\sqrt{v_{th}^3}}$$

$$\sim \frac{k_\theta \rho V}{L_\perp} \equiv \text{diamagnetic frequency}$$

$$V_{dia} \sim \frac{\rho V}{L_\perp} \rightarrow \text{diamagnetic velocity}$$

→ Toward Macroscopics

- To compute anything of relevance (i.e. dispersion relation, wave frequency, etc.), its necessary to compute moments

i.e.
$$\left.\begin{aligned} \hat{n} &= \int d^3v \, f \\ \hat{J}_{\parallel i,e} &= \int d^3v \, q v_\parallel f \end{aligned}\right\} \text{etc.}$$

so as solving GKE obtains g_k in g.c variables $\leftrightarrow$ must

- transform to real/particles variables

$$\Rightarrow \quad g = \sum_{k} g_k e^{i\underline{k}\cdot\underline{x}}$$

$$= \sum_{k} \underbrace{g_k e^{i\underline{k}\cdot\underline{R}}}_{g_{g.c.}} e^{ik_\perp \rho \sin(\phi - \chi)}$$

and, as will seek ϕ integrated quantity

i.e. $n = \int d^3v \, f$ etc. $\quad \int d\phi$

$J_{\parallel} = \int d^3v \, v_{\parallel} f$

$$\Rightarrow \quad g_k = J_0(k_\perp \rho) \, g_k^{g.c.}$$

— Alfven
— int
— DW

For electromagnetic perturbations, note: — Alfven

- magnetic field lines 'wiggle' $\leftrightarrow$ Alfvenic oscillations

$$v_{\parallel}(\hat{n}\cdot\nabla) = v_{\parallel}\left(\partial_z + \frac{\tilde{B}_\perp}{B_0}\cdot\nabla_\perp\right)$$

but, for $\tilde{B}_\perp \gg \tilde{B}_{\parallel}$ (low β)

$$\tilde{B} = \nabla \times (A_{\parallel} \hat{z})$$

$\hookrightarrow$ component of vector potential along $\underline{B_0}$

so

$$v_{\parallel}\hat{n}\cdot\nabla - \frac{c}{B}\nabla\phi\times\hat{z}\cdot\nabla = v_{\parallel}\partial_z - \frac{c}{B_0}\nabla\left(\overset{\mathcal{L}}{\phi} - \frac{v_{\parallel}}{c}\hat{A}_{\parallel}\right)\times\hat{z}\cdot\nabla$$

total $\perp$ scattering

so, in ω_* term $\phi \to \phi - \frac{v_{\parallel}}{c}A_{\parallel}$

— $\hat{E}_{\parallel}$ contains inductive piece

$$\hat{E}_{\parallel} = -\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \nabla_{\parallel}\phi$$

$$\frac{q}{m}E_{\parallel}\frac{\partial f_0}{\partial v_{\parallel}} = \frac{q}{\cancel{m}}\left(+\frac{1}{c}i\omega A_{\parallel\,\underline{k},\omega} - ik_z\phi_{\underline{k},\omega}\right)\frac{-v_{\parallel}}{T/\cancel{m}}\bar{f}_0$$

New — cancels, leaving on RHS $\omega\phi$ term

$$= \text{old} - \frac{i\omega}{c}\frac{q v_{\parallel}}{T}A_{\parallel}\, q\bar{f}_0$$

$$= \omega\left(\phi - \frac{v_{\parallel}A_{\parallel}}{c}\right)\frac{q\bar{f}_0}{T}$$

so $\phi \to \left(\phi - \frac{v_{\parallel}A_{\parallel}}{c}\right)$ in all RHS

$$(\omega - \omega_{d_i} - k_{\parallel}v_{\parallel})\,g_{\underline{k}} = -\frac{i|e|}{T_i}\bar{f}_0\,(\omega - \omega_{*i})\left(\phi - \frac{v_{\parallel}}{c}\hat{A}_{\parallel}\right)_{\underline{k}}J_0(k_{\perp}\rho_i)$$

is linear, e.m. gyrokinetic equation.

Using Gyrokinetics – Kinetic Shear Alfven Wave

Now, $-i(\omega-\omega_{D_i}-k_{\parallel}v_{\parallel_i})\hat{g}_{i\underline{k},\omega} = -\frac{i|e|}{T_i}(\omega-\omega_{*i})\left(\hat{\phi}-\frac{v_{\parallel i}}{c}\hat{A}_{\parallel}\right)_{\underline{k},\omega} \bar{f}_{0_i} J_0$

ions $J_0 = J_0(k_\perp\rho_i)$

$-i(\omega-\omega_{De}-k_{\parallel}v_{\parallel e})\hat{g}_{e\underline{k},\omega} = \frac{i|e|}{T_e}(\omega-\omega_{*e})\left(\hat{\phi}-\frac{v_{\parallel e}}{c}\hat{A}_{\parallel}\right)_{\underline{k},\omega} \bar{f}_{0e}$

electrons

$$\begin{cases} \hat{n}_e = \hat{n}_i & \Rightarrow \text{quasineutrality} \quad (k^2\lambda_D^2 \ll 1) \\ \nabla_\perp^2 \hat{A}_{\parallel} = -\frac{4\pi}{c}\left(\tilde{J}_{\parallel e}+\tilde{J}_{\parallel i}\right) & \Rightarrow \text{Ampere's Law} \end{cases}$$

$\Rightarrow$ macroscopic relations $\Rightarrow$ (shear Alfven)/(reduced MHD) fluctuations!

$$\frac{\hat{n}_e}{n_0} = \frac{|e|}{T_e}\hat{\phi} + \int d^3v\, \hat{g}_e$$

$$\frac{\hat{n}_i}{n_0} = -\frac{|e|}{T_i}\hat{\phi} + \int d^3v\, J_0(k_\perp\rho)\hat{g}_i = +\rho^2\nabla^2\frac{|e|\phi}{T} + \hat{n}_{i\,GC}$$

and basic ordering: $v_{Ti} < \frac{\omega}{k_{\parallel}} < v_{Te}$

QN

For electrons:

$$ik_{\parallel}v_{\parallel e}\hat{g}_{e\underline{k},\omega} - i(\omega-\omega_{De})\hat{g}_{e\underline{k},\omega} = \frac{i|e|}{T_e}(\omega-\omega_{*e})\left(\hat{\phi}-\frac{v_{\parallel}}{c}\hat{A}_{\parallel}\right)_{\underline{k},\omega}\bar{f}_{0e}$$

$v_{\parallel} \gg \omega/k_{\parallel} \Rightarrow$ A's balance;

(adiabatic limit expansion) – like electrons in ion acoustic

$$\hat{g}_{e\underline{k},\omega} = \frac{i|e|}{T_e}\frac{(\omega-\omega_{*e})}{ik_{\parallel}v_{\parallel e}}\left(-\frac{v_{\parallel}}{c}\hat{A}_{\parallel}\right)_{\underline{k},\omega}\bar{f}_{0e}$$

$$\hat{g}_{e\,\underline{k}\,\omega} = \frac{-\cancel{i}|e|}{\cancel{i}\,T_e}\,\frac{\omega-\omega_{*e}}{c k_{\parallel}}\,\hat{A}_{\parallel\,\underline{k}\,\omega}\,\bar{f}_{0e}$$

$$\Rightarrow \frac{\hat{n}_{e\,\underline{k}\,\omega}}{n_0} = \frac{|e|}{T_e}\left[\hat{\phi} - \left(1-\frac{\omega_{*e}}{\omega}\right)\frac{\omega \hat{A}_{\parallel}}{c k_{\parallel}}\right]_{\underline{k},\omega}.$$

For ions: $\omega > k_{\parallel} v_{\parallel i}$ $\rightarrow$ hydrodynamic expansion o/a' ions in ion acoustic

$$-i\left(\omega-\omega_{Di}-k_{\parallel}v_{\parallel i}\right)\hat{g}_{i\,\underline{k}\,\omega} = \frac{-i|e|}{T_i}\left(\omega-\omega_{*i}\right)\left(\hat{\phi}-\frac{v_{\parallel i}}{c}\hat{A}_{\parallel}\right)_{\underline{k}\,\omega}\bar{f}_{0i}J_0(k_\perp\rho)$$

$$\Rightarrow \hat{g}_{i\,\underline{k}\,\omega} = \frac{|e|}{T_i}\left[\frac{(\omega-\omega_{*i})\,\bar{f}_0 J_0(k_\perp\rho)}{(\omega-\omega_{Di}-k_{\parallel}v_{\parallel i})}\right]\left(\hat{\phi}-\frac{v_{\parallel}}{c}\hat{A}_{\parallel}\right)_{\underline{k},\omega}$$

Now: $\rightarrow \omega > k_{\parallel} v_{\parallel i}$ $\quad \omega > \omega_{Di}$

$\Rightarrow$ fluid ions

$\rightarrow$ $\bar{f}_{0i}$ even $\rightarrow$ $\hat{A}_{\parallel}$ term cancels

$\rightarrow k_\perp\rho_i \ll 1$

$$\Rightarrow \hat{g}_{i\,\underline{k}\,\omega} J_0(k_\perp\rho_i) = \frac{|e|}{T_i}\frac{(\omega-\omega_{*i})}{\omega}\bar{f}_0 J_0^2(k_\perp\rho_i)\,\hat{\phi}_{\underline{k}\,\omega}$$

$$= \frac{|e|}{T_i}\hat{\phi}_{\underline{k}\,\omega}\left(1-\frac{\omega_{*i}}{\omega}\right)\bar{f}_0 J_0^2(k_\perp\rho_i)$$

$$\frac{\hat{n}_i}{n_0} = -\frac{|e|}{T_i}\hat{\phi}_{\underline{k}\,\omega}\left[1-\int d^3v\,\bar{f}_0 J_0^2(k_\perp\rho_i)\left(1-\frac{\omega_{*i}}{\omega}\right)\right]$$

$$\int d^3v\, \bar{f}_0\, \bar{J}_0^2(k_\perp \rho_i) = I_0(b_i)\, e^{-b_i}$$

$$b_i \equiv k_\perp^2 \rho_{i_T}^2$$

$$\therefore \quad \frac{\hat{n}_i}{n_0} = -\frac{|e|}{T_i}\,\hat{\phi}_{\underline{k},\omega}\left[1 - \left(1 - \frac{\omega_{*i}}{\omega}\right) I_0(b_i)\, e^{-b_i}\right]$$

expanding in $b_i \ll 1$,

$$\frac{\hat{n}_i}{n_0} \cong -\frac{|e|}{T_i}\,\hat{\phi}_{\underline{k},\omega}\left[1 - \left(1 - \frac{\omega_{*i}}{\omega}\right)(1 - b_i)\right]$$

Now, in MHD ordering: $\omega > \omega_{*i}, \omega_{*e}$ $\qquad \cancel{1} - \cancel{1} + b_i + \frac{\omega_{*i}}{\omega}$

$\Rightarrow$

$$\frac{|e|}{T_e}\left[\hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel}\right]_{\underline{k},\omega} = -\frac{|e|}{T_i}\,\hat{\phi}_{\underline{k},\omega}\left[b_i\right] \qquad \rho_s^2 \equiv c_s^2/\Omega_i^2$$

$$= -k_\perp^2 \rho_s^2\, \frac{|e|}{T_e}\,\hat{\phi}_{\underline{k},\omega}$$

$$\therefore \quad \left[\hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel}\right]_{\underline{k},\omega} = -k_\perp^2 \rho_s^2\, \hat{\phi}_{\underline{k},\omega}$$

Note:

- LHS $\sim \hat{E}_{\parallel \underline{k},\omega}/k_\parallel$ $\qquad E_\parallel = -\nabla_\parallel \phi - \frac{1}{c}\frac{\partial A}{\partial t}$

$\therefore$ QN $\Rightarrow$ $\hat{E}_{\parallel \underline{k},\omega} \sim +k_\parallel k_\perp^2 \rho_s^2\, \hat{\phi}_{\underline{k},\omega}$ $\quad \rightarrow$ kinetic Alfven

$\Rightarrow$ for $k_\perp^2 \rho_s^2 \ll 1 \Rightarrow \hat{E}_\parallel \cong 0$

in MHD (ideal): $\underline{E} + \frac{\underline{v} \times \underline{B}}{c} \cong 0$

$$\hat{n} \cdot \left[\underline{E} + \frac{\underline{v} \times \underline{B}}{c} \right] = 0 \Rightarrow \underline{\hat{E}_\parallel = 0}$$

$\therefore$ QN $\Leftrightarrow$ effective 'Ohm's Law'

- content $\left(\hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel} \right)_{\substack{k \\ \omega}} = -k_\perp^2 \rho_s^2 \hat{\phi}_{\substack{k \\ \omega}}$

$\sim \hat{E}_\parallel$ polarization drift (ions only)

i.e. $\frac{\tilde{n}_e}{n} \sim \hat{E}_\parallel$ (electron flow along field) balances $\frac{\hat{n}_i}{n_0} \sim -k_\perp^2 \rho^2 \hat{\phi}$,

as $E \times B$ drifts cancel.

<u>Ampere's Law</u> $\nabla \cdot J = 0$

- Now, for 2^{nd} equation, can compute $\hat{J}_{\parallel e}, \hat{J}_{\parallel i}$ and plug into Ampere's Law

but can notice, from basic equations:

$\int d^3v\, J_0(k_\perp \rho) \hat{g}^i e_z n - \int d^3v\, \hat{g}^e e_z n \Rightarrow$ forms current!

$$\textcircled{1} \Rightarrow J_0(k_\perp\rho_i)\left(-i(\omega-\omega_{di}-k_\parallel v_{\parallel i})\hat{g}_i\right) = -\frac{i|e|}{T_i}(\omega-\omega_{*i})J_0^2(k_\perp\rho_i)\overline{f_0}\left(\hat{\phi}-\frac{v_\parallel}{c}\hat{A}_\parallel\right)_{k}$$

$$\textcircled{2}\quad -i(\omega-\omega_{de}-k_\parallel v_{\parallel e})\hat{g}_e = \frac{i|e|}{T_e}(\omega-\omega_{*e})\left(\hat{\phi}-\frac{v_\parallel}{c}\hat{A}_\parallel\right)_{k}\overline{f_0}$$

$$\int d^3v_i\ \textcircled{1} - \int d^3v_e\ \textcircled{2} \Rightarrow$$

$$-i\omega\left[\int d^3v\, J_0(k_\perp\rho_i)\hat{g}_i - \int d^3v\,\hat{g}_e\right] + i k_\parallel \frac{\hat{J}_\parallel}{n_0 q} \qquad \nabla\cdot J = 0$$

$$+i\left[\int d^3v\,\omega_{di}\hat{g}_i J_0(k_\perp\rho_i) - \int d^3v\,\omega_{de}\hat{g}_e\right]$$

$$= -\frac{i|e|}{T_i}(\omega-\omega_{*i})\left(\int d^3v\,\overline{f_0}J_0^2(k_\perp\rho_i)\right)\hat{\phi}_k$$

$$-\frac{i|e|}{T_e}(\omega-\omega_{*e})\,\hat{\phi}_{k,\omega} \qquad \text{by even } \overline{f_0} \text{ in } v_\parallel\,.$$

Now, $\textcircled{1}$ QN $\Rightarrow$

$$-\frac{|e|\hat{\phi}}{T_i} + \int d^3v\, J_0(k_\perp\rho_i)\,\hat{g}_i$$

$$\doteq \frac{|e|\hat{\phi}}{T_e} + \int d^3v\,\hat{g}_e$$

$$\textcircled{2}\int d^3v\,\overline{f_0}\,J_0^2 = I_0(b_i)\,e^{-b_i}$$

$$\cong (1-k_\perp^2\rho_i^2) \qquad k_\perp^2\rho_i^2 < 1$$

$$\Rightarrow \frac{i k_\parallel \hat{J}_\parallel}{n_0 q} + i\left[\int d^3v\, \omega_{di}\, \hat{g}_{i_k} J_0(k_\perp \rho_i) - \int d^3v\, \omega_{de}\, \hat{g}_{e_k}\right]$$

$$= +\frac{i|e|}{T_i}\, \omega\, k_\perp^2 \rho_i^2\, \hat{\phi}_k + i|e|\left(\frac{\omega_{*i}}{T_i} I_0(b_i) e^{-b_i} + \frac{\omega_{*e}}{T_e}\right)\hat{\phi}_k$$

but further:

$$= \frac{i|e|}{T_i}\, \omega\, k_\perp^2 \rho_i^2\, \hat{\phi}_k + i|e|\left(\frac{\omega_*}{T_i}(\cancel{1}-b_i) \cancel{+ \frac{\omega_{*e}}{T_e}}\right)\hat{\phi}_k$$

$$\frac{\omega_{*i}}{T_i} = -\frac{\omega_{*e}}{T_e}$$

(species $E\times B$ drift together)

Now;

$$\boxed{\frac{i k_\parallel J_\parallel}{n_0|e|} + i\left[\int d^3v\, \omega_{di}\, \hat{g}_{i_k} J_0(k_\perp \rho_i) - \int d^3v\, \omega_{de}\, \hat{g}_{e_k}\right] = \frac{i|e|\hat{\phi}_k}{T_i}(\omega - \omega_{*i})\, k_\perp^2 \rho_i^2}$$

① ② ③ (5.2)

- equivalent to $\nabla\cdot J = 0$

i.e. $\frac{\partial\hat{\rho}}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0 \qquad \frac{\partial\hat{\rho}}{\partial t} = -i\omega\left(\hat{n}_i|e| - \hat{n}_e|e|\right)$

$= 0$ for $k^2\lambda_D^2 < 1$

$\therefore \nabla\cdot J = 0$

$\Rightarrow \nabla_\parallel J_\parallel = -\underline{\nabla}_\perp\cdot\underline{\hat{J}}_\perp$

$$\hat{J}_\perp = \cancel{J_i^{E\times B}} - \cancel{J_e^{E\times B}} + J_i^{pol} + J^{drift}$$

E×B drift of species equal

$\therefore$ ~~Ø~~ $k_\parallel \hat{J}_\parallel \rightarrow \nabla_\parallel J_\parallel$

(3) $\rightarrow \nabla_\perp \cdot J_{\perp\,pol}$.

(2) $\rightarrow \nabla_\perp \cdot J_\perp^{drift}$.

$\Rightarrow$ quite general expression for $k_\perp \rho_i < 1$. Can evaluate for various cases.

Thus have gyrokinetic moment equations for $k_\perp^2 \rho_i^2 < 1 \Rightarrow$
$\omega > \omega_{*i}$ (MHD)

$$\hat{\phi}_{\underline{k}} - \frac{\omega \hat{A}_{\parallel \underline{k}}}{c k_\parallel} = -k_\perp^2 \rho_s^2 \hat{\phi}_{\underline{k}}$$

$$\frac{i k_\parallel \hat{J}_{\parallel \underline{k}}}{n_0 |e|} + i \left[\int d^3v\, \omega_{di} \hat{g}_{i\underline{k}}\, J_0(k_\perp \rho_i) - \int d^3v\, \omega_{de} \hat{g}_{e\underline{k}} \right]$$

$$= \frac{i|e|}{T_e} k_\perp^2 \rho_s^2\, \omega \hat{\phi}_{\underline{k}}$$

Now, for basic excitation/wave: Kinetic Shear Alfven Wave!

$$\hat{J}_{\parallel \underline{k}} = \frac{+c}{4\pi} k_\perp^2 \hat{A}_{\parallel \underline{k}} \quad ; \quad (1 + k_\perp^2 \rho_s^2) \hat{\phi}_{\underline{k}} = \frac{\omega}{c k_\parallel} \hat{A}_{\parallel \underline{k}}$$

⇔ and, furthermore $\omega_d \to 0$ (straight field or/and $\omega \gg \omega_d$)

$$\frac{\cancel{i} k_\parallel}{n_0|e|}\frac{c}{4\pi}\cancel{k_\perp^2}\frac{c\,k_\parallel}{\omega}\frac{I(\,)}{|e|}\frac{|e|\cancel{\hat\phi}}{T_e} = \cancel{i}\,\frac{|e|\cancel{\hat\phi}}{T_e}\cancel{k_\perp^2}\rho_s^2\,\omega$$

$$\frac{c^2 m_i^2}{|e|^2 B_0^2}\frac{T_e}{m_i}\left(\frac{B_0^2}{4\pi n_0 m_i}\right)\frac{k_\parallel^2}{\omega}\left(1+k_\perp^2\rho_s^2\right) = \omega\,\rho_s^2$$

$$\cancel{\rho_s^2}\,V_A^2 k_\parallel^2\left(1+k_\perp^2\rho_s^2\right) = \omega^2\,\cancel{\rho_s^2}$$

⇒ KSAW dispersion relation:

$$\boxed{\omega^2 = k_\parallel^2 V_A^2\left(1+\underbrace{k_\perp^2\rho_s^2}\right)}$$

→ cross-field propagation from polarization drift.

or

$$\boxed{k_\perp^2 = \left(\frac{\omega^2}{k_\parallel^2 V_A^2}-1\right)\Big/\rho_s^2}$$

- $V_A^2 = \left(B_0^2/4\pi\right)\left(n_0 m_i\right)$ → analogue to waves on string

magnetic tension

ion inertia

$\mu \to n_0 m_i$

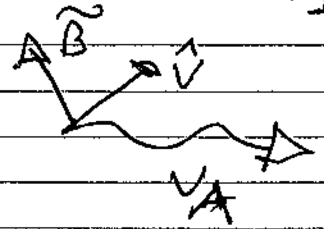
$T \to B^2/4\pi$

→ Note: $\hat\phi - \frac{\omega}{c}\frac{\hat A_\parallel}{k_\parallel} \cong 0$ $\qquad \hat\phi = \frac{V_A}{c}\hat A_\parallel$

$\hat\phi - \frac{k_\parallel V_A}{c\,k_\parallel}\hat A_\parallel \cong 0$ $\qquad$ i.e. $\hat\phi = \hat A_\parallel$, within factor.

- wave propagates along $\underline{B_0}$!

$$\begin{cases} \text{fluid motion} & \underline{V} = -\dfrac{\nabla\phi \times \hat{z}}{B} \quad \perp B_0 \\ \text{magnetic perturbation} & \delta B_\perp \sim \underline{V}, \quad \perp B_0 \end{cases}$$



- note can re-write as:

$$\hat{E}_\parallel = \hat{\phi} - \frac{\omega \hat{A}_\parallel}{c k_\parallel} = 0$$

$$-i\nabla_\parallel \phi = \frac{\omega \hat{A}_\parallel}{c} \Rightarrow \frac{1}{c}\frac{\partial \hat{A}_\parallel}{\partial t} = -\nabla_\parallel \phi$$ (ohms Law)

and

$$\frac{\nabla_\parallel \hat{J}_\parallel}{n_0 |e|} = +\rho_s^2 \frac{\partial}{\partial t} \nabla_\perp^2 \frac{|e|\hat{\phi}}{T_e}$$

$(\underline{\nabla} \cdot \underline{J} = 0)$

$\Rightarrow$ corresponds to linearized reduced MHD!

i.e. upon re-scale:

$$\frac{d}{dt} \nabla_\perp^2 \phi = \nabla_\parallel \hat{J}_\parallel \qquad J_\parallel = \nabla_\perp^2 A_\parallel$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{V}_E \cdot \underline{\nabla}$$

and $\frac{\partial A}{\partial t} + \underline{V}_E \cdot \underline{\nabla} A = -\nabla_{\parallel} \hat{\phi}$.

bottom Line : Reduced MHD $\Leftrightarrow$ Gyrokinetic Ordering.
$\Updownarrow$
incompr. MHD