

Physics 161: Black Holes: Lecture 12: 1 Feb 2010

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12 Orbits in the Schwarzschild metric continued

12.1 Effective Potential for Schwarzschild Orbits

Let's apply the same effective potential technique to the full r geodesic equation we derived earlier.

$$m \left(\frac{dr}{d\tau} \right)^2 = \frac{E^2}{mc^2} - \left(1 - \frac{r_S}{r} \right) \left(mc^2 + \frac{l^2}{mr^2} \right),$$

where $r_S = 2GM/c^2$. Here the effective potential is

$$V_{eff} = \left(1 - \frac{r_S}{r} \right) \left(mc^2 + \frac{l^2}{mr^2} \right),$$

and keep in mind that we are using the proper time τ , not t as the time variable, and l is the angular momentum and E is the total energy. We can spot the different types of orbits as we did for the Newtonian effective potential. Now we see there are five different types of orbits.

Fig: Effective potential for Schwarzschild Geodesics

1. At the minimum we have a bound circular orbit. The radius is found from $dV_{eff}/dr = 0$ as before.
2. We also have the bound orbits with turning points r_1 and r_2 like in the Newtonian case, but here these turn out to not be elliptical! The orbits don't close. If we do the perihelion advance of Mercury we will show this. The time to between r_1 and r_2 and back is more than the time to go around $\phi = 360^\circ$, so the perihelion advances.
3. Next, as for Newtonian hyperbolic orbits, we have orbits that start at $r = \infty$, come in, reach a distance of closest approach, and go out again. Close examination shows these are close to, but not exactly the same as, the Newtonian case.
4. Now look at maximum point on top of the hill. We find this point also when we set $dV_{eff}/dr = 0$, since the slope is zero here. If you set a particle there and carefully balanced it, it would stay there. Thus this is another circular orbit, but it is unstable. A tiny perturbation outward and the particle will escape to infinity. A tiny perturbation inward and it will fall in the hole.
5. Finally, there is another new type of orbit not found in the Newtonian case, the capture orbit. If the energy is high enough, the particle goes over the high point and then plunges down into the

hole, never to return. In the Newtonian case a particle can never be captured in this way due to gravity alone, even though this is how we think when, for example, a comet hits the Sun. Hitting the Sun's surface invokes non-gravitational forces; a point mass object could never capture anything in Newtonian mechanics.

Let's find the radii of the circular orbits. We can write

$$V_{eff} = m + \frac{l^2}{mr^2} - \frac{r_S m}{r} - \frac{r_S l^2}{mr^3},$$

and differentiate and set to zero.

$$\frac{dV_{eff}}{dr} = -\frac{2l^2}{mr^3} + \frac{r_S m}{r^2} + \frac{3r_S l^2}{mr^4} = 0.$$

We can simplify this to a quadratic equation in r by multiplying through by mr^4 :

$$r_S m^2 c^2 r^2 - 2l^2 r + 3r_S l^2 = 0,$$

which has two solutions:

$$r_{\pm} = \frac{l^2}{r_S m^2 c^2} \left(1 \pm \sqrt{1 - \frac{3r_S^2 m^2 c^2}{l^2}} \right),$$

where I've put the c 's back in for fun and reference.

Note that if the quantity in the square root is negative we don't have a real solution. Thus there is no circular orbit in that case. This happens when the angular momentum is too small, i.e. there are only capture orbits. If the quantity in the square root is positive, then we have two solutions, that is, two circular orbits at different radii, as we saw in the plot. Thus we have two circular orbit solutions if $l^2 > 3r_S^2 m^2 c^2$, and none if not.

When there are two orbits we can tell if the orbits are stable or unstable by taking the 2nd derivative. If $d^2 V_{eff}/dr^2 > 0$, the the curvature of the effective potential is positive and it is a minimum. This means the orbit is stable. On the other hand if $d^2 V_{eff}/dr^2 < 0$, it means there is a maximum in the effective potential at that point, and the orbit is unstable. We can do the math, but we saw before from the picture that the smaller solution (the one with the minus square root) was unstable, and the larger solution with the plus square is a stable orbit.

From the above discussion we see that the smallest possible *stable* circular orbit will happen when the square root term vanishes. This happens when $1 - 3r_S^2 m^2 c^2/l^2 = 0$, or $l^2 = 3r_S^2 m^2 c^2$. The value of the radius for this value of the angular momentum can be found just by plugging this value of l into the formula for r_{\pm} , and we find that the minimum stable circular radius is

$$r_{min} = \frac{l^2}{r_S m^2 c^2} = 3r_S,$$

that is just exactly three times the Schwarzschild radius. Actually we should check that this orbit is stable, since for this value of l^2 , the stable and unstable orbits are the same. Plugging in r_{min} , and the value of l^2 above into

$$\frac{d^2 V_{eff}}{dr^2} = \frac{6l^2}{mr^4} - \frac{2r_S m}{r^3} - \frac{12r_S l^2}{mr^5},$$

we find that $d^2V_{eff}/dr = 0$ at this combined minimum. Thus this orbit is just really neither stable nor unstable, but neutral. It is called, however, the minimum or last stable orbit, since an orbit just a tiny bit larger is in fact stable.

This last stable orbit value is a very important result. When things fall into black holes they have trouble getting in because of angular momentum. They tend to get ripped apart and form what are called accretion disks. The material in the disk gradually spirals inward. However, when the radius reaches $3r_S$, there is no longer a stable circular orbit, so the material all just flows into the hole. Thus we expect that when we look at real black holes in space we will see material down to $3r_S$ but not any closer. When we detect X-rays from black holes, we hope we are looking at material radiating from the distance $3r_S$. And when we find periodic signals from black hole candidates, we assume this is the radius at which the material is orbiting.

12.2 Producing energy using black holes

Now let me tell you why black holes are the most efficient energy generation devices we have yet conceived. Much better than nuclear fission or even nuclear fusion. Of course, there is the small problem of not having a black hole near by and available for use.

First consider throwing some junk into a black hole from far away. We learned before that the total energy of the stuff far from the black hole is just $E = mc^2$, where m is the mass of the stuff. Suppose this stuff doesn't go straight into the hole, but is slowed down in a series of orbits, finally settling into the last stable orbit we just discussed. What is the energy of the stuff now? We can use the radial geodesic equation to find out. On a circular orbit $dr/d\tau = 0$, so we have

$$\frac{E^2}{m} = \left(1 - \frac{r_S}{r}\right) \left(m + \frac{l^2}{mr^2}\right).$$

Next we plug in $l^2 = 3r_S^2m^2$, and $r = 3r_S$,

$$\frac{E^2}{m} = \left(1 - \frac{1}{3}\right) \left(m + \frac{3r_S^2m^2}{m9r_S^2}\right) = \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{3}\right)m = \frac{8}{9}m.$$

Thus

$$E = \sqrt{\frac{8}{9}}mc^2.$$

Somehow during the transition from far away to the last stable orbit the energy has decreased. That energy must have been radiated away, and therefore be available for other use. The amount of energy radiated away is $(mc^2 - E)/mc^2 = 1 - \sqrt{\frac{8}{9}} = 5.7\%$.

Let's compare this with other forms of energy generation. When burned, a gallon of gasoline produces 1.32×10^8 Joules of energy and has a mass of 2.8 kg. Thus the fractional mass energy lost when a gallon of gasoline is burned is $\Delta E/E = (1.32 \times 10^8)/((2.8)(3 \times 10^8)^2) = 5.2 \times 10^{-10}$. This is thus also the fraction of the rest mass of the gasoline that is turned into useful energy, a far cry from the 5.7% we get from the black hole.

Next, consider nuclear energy, say the complete fission of 1 kg of pure Uranium 235. This produces 8.28×10^{13} Joules, so $\Delta E/E = (8.28 \times 10^{13})/((1)(3 \times 10^8)^2) = 0.0092\%$, still below the black hole. Finally one usually thinks of the main source of energy in the Universe as nuclear fusion of Hydrogen into Helium (as in the Sun). Actually in large stars, the Helium is then burned to Carbon, Oxygen, etc. all the way

up to Iron, which is the most stable element. This whole chain of nuclear fusion releases about 0.9% of the rest mass into usable energy, still a factor of six below what you get from throwing your garbage into a black hole. Black holes are the best devices we know of for producing energy. In fact, using a rotating black hole we can turn even more of the rest mass of stuff into useful energy. The energy generation method discussed here is thought to be the power source of quasars, the most luminous and energetic steady objects in the Universe.