Homework 6 – Solutions

1. (a) $U_0 = -I m_s^2 \times$ the total # of nearest-neighbor pairs in the system

$$= -I m_s^2 \times \frac{1}{2} N q .$$

It follows that

(4) $$C_0 = -I \cdot 2 m_s \frac{d m_s}{dT} \times \frac{1}{2} N q = -N I q \, m_s \frac{d m_s}{dT} .$$

(b) As $T \to T_c$ from below, $m_s \simeq \sqrt{3\left(1 - \frac{T}{T_c}\right)}$.

$$\therefore \; C_0 \longrightarrow -N I q \sqrt{3} \left(1 - \frac{T}{T_c}\right)^{1/2} \cdot \sqrt{3} \cdot \frac{1}{2} \left(1 - \frac{T}{T_c}\right)^{-1/2} \times \left(-\frac{1}{T_c}\right)$$

$$= N I q \cdot \frac{3}{2} \cdot \frac{1}{T_c}$$

(4) $$= \frac{3}{2} N k \quad \left[\because \; I q = k T_c \right].$$

2. The given equation of state may be written as

$$\mu_B H + Iq m = kT \tanh^{-1} m, \tag{1}$$

so that

$$\mu_B \, dH + Iq \, dm = k \tanh^{-1} m \; dT + kT \frac{1}{1-m^2} dm.$$

It follows that

$$\left(\frac{\partial m}{\partial H}\right)_T = \frac{\mu_B}{\frac{kT}{1-m^2} - Iq} = \frac{\mu_B}{k} \frac{1-m^2}{T-(1-m^2)T_c},$$

and

$$\left(\frac{\partial m}{\partial T}\right)_H = \frac{k \tanh^{-1} m}{Iq - \frac{kT}{1-m^2}} = \frac{(1-m^2)\tanh^{-1} m}{(1-m^2)T_c - T}.$$

We thus get

5 $$\chi = N\mu_B \left(\frac{\partial m}{\partial H}\right)_T = \frac{N\mu_B^2}{k} \frac{1-m^2}{T-(1-m^2)T_c}, \tag{2}$$ ✓

and

$$C_H - C_M = \frac{N^2 \mu_B^2 T}{\chi} \left(\frac{\partial m}{\partial T}\right)_H^2$$

$$= N^2 \mu_B^2 T \frac{k\left[T-(1-m^2)T_c\right]}{N\mu_B^2(1-m^2)} \left[\frac{(1-m^2)\tanh^{-1} m}{(1-m^2)T_c - T}\right]^2$$

5 $$= Nk \frac{T(1-m^2)\left(\tanh^{-1} m\right)^2}{T-(1-m^2)T_c}. \tag{3}$$ ✓

Note that this last result could also be obtained by using the formula

$$C_H - C_M = -T\left(\frac{\partial H}{\partial T}\right)\left(\frac{\partial M}{\partial T}\right),$$

with

$$\left(\frac{\partial H}{\partial T}\right)_M = \frac{k}{\mu_B} \tanh^{-1} m \qquad \& \qquad M = N\mu_B m.$$

(a) The "Curie regime" corresponds to $H > 0$ and T such that $m \ll 1$. Eqns. (1) – (3) then give

$$m \simeq \frac{\mu_B H + Iqm}{kT} = \frac{\mu_B H}{kT} + \frac{T_c}{T} m, \text{ so that}$$

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$$m \simeq \frac{\mu_B H}{k(T-T_c)} \quad \text{or} \quad M \simeq \frac{N\mu_B^2 H}{k(T-T_c)}, \tag{4}$$

$$\chi \simeq \frac{N\mu_B^2}{k(T-T_c)}, \tag{5}$$

and

$$C_H - C_M \simeq Nk \frac{Tm^2}{T-T_c} \simeq \frac{N\mu_B^2}{k} \frac{TH^2}{(T-T_c)^3}. \tag{6}$$

You may extract from here expressions for the paramagnetic case by putting $T_c = 0$ and noting that the quantity $N\mu_B^2/k$ is nothing but the Curie constant for the model under study!

(b) In the limit $H \to 0$, $m \to m_s$. Now, if $T > T_c$, then $m_s = 0$ and we get from (2) and (3)

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$$\chi = \frac{N\mu_B^2}{k(T-T_c)} \quad \& \quad C_H - C_M = 0. \tag{7,8}$$

However, if $T \lesssim T_c$, then $|m_s| \ll 1$ and eqn. (1) gives

$$m_s = \frac{T}{T_c}\left(m_s + \frac{1}{3}m_s^3 + \cdots\right), \text{ so that}$$

$$m_s^2 \simeq 3\left(1 - \frac{T}{T_c}\right).$$

We now get from (2) and (3)

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$$\chi \simeq \frac{N\mu_B^2}{k}\,\frac{1}{m_s^2 T_c - (T_c - T)}$$

$$= \frac{N\mu_B^2}{k}\,\frac{1}{2(T_c - T)}, \qquad (9)$$

and

$$C_H - C_M \simeq Nk\,\frac{T_c \cdot m_s^2}{m_s^2 T_c - (T_c - T)}$$

$$= \frac{3}{2}Nk. \qquad (10)$$

3. (a) It was proved in the class that

$$C_H - C_M = -T\left(\frac{\partial H}{\partial T}\right)_M \left(\frac{\partial M}{\partial T}\right)_H . \qquad (1)$$

Using the cyclic rule, we have

$$\left(\frac{\partial H}{\partial T}\right)_M = -\left(\frac{\partial H}{\partial M}\right)_T \left(\frac{\partial M}{\partial T}\right)_H .$$

Eqn. (1) then becomes

$$C_H - C_M = T\left(\frac{\partial H}{\partial M}\right)_T \left[\left(\frac{\partial M}{\partial T}\right)_H\right]^2 .$$

Since $\chi = (\partial M/\partial H)_T$ and $M = N\mu_B m$, we get

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$$C_H - C_M = \left(N^2 \mu_B^2 T/\chi\right)\left[(dm/dT)_H\right]^2 .$$

(b) As $T \to T_c$ from below,

$$m \longrightarrow \sqrt{3\left(1-\frac{T}{T_c}\right)} \quad \& \quad \chi_T \longrightarrow \frac{N\mu_B^2}{2k(T_c - T)} .$$

It follows that

$$(C_H - C_M) \longrightarrow \frac{N^2 \mu_B^2 T_c}{\left[N\mu_B^2/2k(T_c - T)\right]} \left\{\frac{\sqrt{3}}{2}\left(1-\frac{T}{T_c}\right)^{-1/2} \frac{-1}{T_c}\right\}^2$$

$$= NT_c \cdot 2k(T_c - T) \cdot \frac{3}{4}\left(\frac{T_c - T}{T_c}\right)^{-1} \frac{1}{T_c^2}$$

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$$= \frac{3}{2} Nk.$$