HW-4 Solutions

1. For black-body radiation,

$$U = aVT^4, \quad S = \frac{4}{3}aVT^3 \text{ and } P = \frac{1}{3}aT^4,$$

where a is a universal constant. It follows that

$$U = 3PV \quad \text{and} \quad S = 4PV/T.$$

(a)

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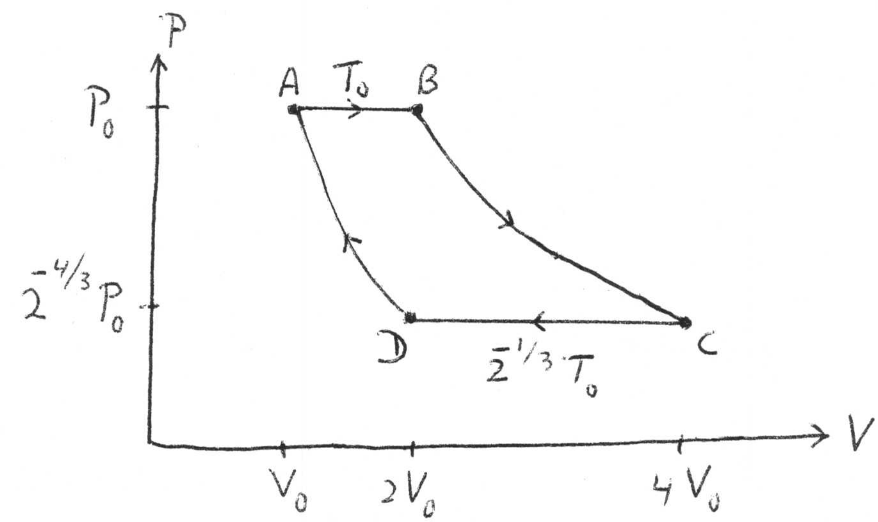

(b) Process AB: $\quad T = T_0, \; P = P_0, \; \Delta V = 2V_0 - V_0 = V_0.$

$$\Delta U = 3P_0\,\Delta V = 3P_0V_0, \text{ and} \tag{1}$$

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$$Q_{AB} = T_0\,\Delta S = T_0 \cdot (4P_0/T_0)\,\Delta V = 4P_0V_0. \tag{2}$$

$$\therefore \quad W_{AB} = Q_{AB} - \Delta U = P_0V_0. \tag{3}$$

Process BC: $\quad S = \text{const.} \;\therefore\; T \propto V^{-1/3}$ and hence $P \propto V^{-4/3}$.

It follows that $P_C = 2^{-4/3}P_0$ and $T_C = 2^{-1/3}T_0$.

Now $\quad Q_{BC} = 0$, and (4)

$$\Delta U = 3\left(2^{-4/3} P_0 \cdot 4V_0 - P_0 \cdot 2V_0\right)$$

(3)
$$= -6 P_0 V_0 \left(1 - 2^{-1/3}\right) < 0. \qquad (5)$$

$$\therefore \quad W_{BC} = Q_{BC} - \Delta U = 6 P_0 V_0 \left(1 - 2^{-1/3}\right) > 0. \qquad (6)$$

<u>Process CD:</u> $\quad T = 2^{-1/3} T_0, \quad P = 2^{-4/3} P_0, \quad \Delta V = 2V_0 - 4V_0 = -2V_0$

$$\Delta U = 3 \cdot 2^{-4/3} P_0 \cdot (-2V_0) = -3 \cdot 2^{-1/3} P_0 V_0. \qquad (7)$$

$$Q_{CD} = 2^{-1/3} T_0 \cdot \Delta S = 2^{-1/3} T_0 \cdot 4\left(2^{-4/3} P_0 / 2^{-1/3} T_0\right) \cdot \Delta V$$

(3)
$$= -4 \cdot 2^{-1/3} P_0 V_0. \qquad (8)$$

$$\therefore \quad W_{CD} = Q_{CD} - \Delta U = -2^{-1/3} P_0 V_0. \qquad (9)$$

<u>Process DA:</u>

$$Q_{DA} = 0, \qquad (10)$$

$$\Delta U = 3\left(P_0 V_0 - 2^{-4/3} P_0 \cdot 2V_0\right)$$

(3)
$$= 3 P_0 V_0 \left(1 - 2^{-1/3}\right) > 0. \qquad (11)$$

$$\therefore \quad W_{DA} = Q_{DA} - \Delta U = -3 P_0 V_0 \left(1 - 2^{-1/3}\right) < 0. \qquad (12)$$

(c) The <u>net</u> work done is given by the sum of expressions (3), (6), (9) and (12), with the result

$$W_{net} = 4 P_0 V_0 \left(1 - 2^{-1/3}\right). \qquad (13)$$

3 Similarly, the net heat absorbed is given by the sum of expressions (2), (4), (8) and (10), with the result

$$Q_{net} = 4 P_0 V_0 (1 - 2^{-1/3}). \qquad (14)$$

W_{net} is indeed equal to Q_{net} --- because $(\Delta U)_{net} = 0$.

(d) The efficiency η is given by expressions (2) and (13), with the result

3 $$\eta = \frac{4 P_0 V_0 (1 - 2^{-1/3})}{4 P_0 V_0} = 1 - 2^{-1/3} = 1 - \frac{T_{CD}}{T_{AB}},$$

in conformity with the Carnot theorem.

2. An indefinite N implies that the chem. pot. of this gas is zero.

$$\therefore \quad U = \int_0^\infty \frac{\varepsilon\, g(\varepsilon)\, d\varepsilon}{e^{\varepsilon/kT} - 1}.$$

$g(\varepsilon)$ may be obtained by using the fact that $p = \left(\frac{\varepsilon}{a}\right)^{2/3}$ and employing the phase-space expression

$$\int' \frac{dx\, dy\, dp_x\, dp_y}{h^2} = A \, \frac{2\pi p\, dp}{h^2}$$

$$= \frac{2\pi A}{h^2} \left(\frac{\varepsilon}{a}\right)^{2/3} \cdot \left(\frac{1}{a}\right)^{2/3} \frac{2}{3} \varepsilon^{-1/3}\, d\varepsilon$$

$$= \frac{4\pi A}{3h^2 \cdot a^{4/3}} \varepsilon^{1/3}\, d\varepsilon.$$

$$\text{Thus,} \quad \frac{U}{A} = \frac{4\pi}{3h^2 a^{4/3}} \underbrace{\int_0^\infty \frac{\varepsilon^{4/3}\, d\varepsilon}{e^{\varepsilon/kT} - 1)}}_{= (kT)^{7/3}\, \Gamma(7/3)\, \zeta(7/3)}$$

10 $\qquad \propto T^{7/3}.$ ✓

3. For fermions, $N_\epsilon = 0$ or 1.

If $p_\epsilon(n)$ denotes the probability that the quantum state ϵ contains exactly n particles, then

$$\overline{N_\epsilon} = \sum_n p_\epsilon(n) \cdot n = p_\epsilon(0) \cdot 0 + p_\epsilon(1) \cdot 1 = p_\epsilon(1).$$

Thus $\underline{p_\epsilon(1) = \overline{N}_\epsilon}$ & hence $\underline{p_\epsilon(0) = 1 - \overline{N}_\epsilon}$. $\longleftarrow$

In view of the fact that $\overline{N_\epsilon} = \dfrac{1}{e^{(\epsilon-\mu)/kT} + 1}$,

$$p_{\mu+\Delta}(1) = \overline{N}_{\mu+\Delta} = \frac{1}{e^{\Delta/kT} + 1}, \text{ and}$$

$$p_{\mu-\Delta}(0) = 1 - \overline{N}_{\mu-\Delta} = 1 - \frac{1}{e^{-\Delta/kT} + 1}$$

$$= \frac{e^{-\Delta/kT} \cancel{+1} \cancel{-1}}{e^{-\Delta/kT} + 1}$$

$$= \frac{1}{1 + e^{\Delta/kT}}.$$

8 Hence the result!

4. $$\left(\frac{\partial C_H}{\partial H}\right)_T = \left[\frac{\partial}{\partial H}\left\{T\left(\frac{\partial S}{\partial T}\right)_H\right\}\right]_T$$

$$= T\,\frac{\partial^2 S}{\partial H\,\partial T} = T\,\frac{\partial^2 S}{\partial T\,\partial H}.$$

Using (M8), we get

4 $$T\left[\frac{\partial}{\partial T}\left(\frac{\partial M}{\partial T}\right)_H\right]_H = T\left(\frac{\partial^2 M}{\partial T^2}\right)_H.$$ ✓

For a paramagnetic material in the Curie regime,

$M = \dfrac{CH}{T}$. Therefore, $\left(\dfrac{\partial^2 M}{\partial T^2}\right)_H = \dfrac{2CH}{T^3}$.

It follows that

$$\left(\frac{\partial C_H}{\partial H}\right)_T = \frac{2CH}{T^2}.$$

Integrating over H, keeping T const., we get

4 $$C_H = \frac{CH^2}{T^2} + \text{a const. independent of } H,$$ ✓

but possibly a function of T.

Hence the result.

5.

(a) $N_\epsilon \propto e^{-\epsilon/kT}$ (M-B distribution)

$$\therefore \quad \frac{N_\epsilon}{N} = \frac{e^{-\epsilon/kT}}{\sum_\epsilon e^{-\epsilon/kT}} .$$

In the present case, we have only <u>two</u> possibilities:

$\uparrow$ (with $\epsilon = -\mu_B H$) AND $\downarrow$ (with $\epsilon = +\mu_B H$).

Hence $$N_\uparrow = N \frac{e^{\mu_B H/kT}}{e^{\mu_B H/kT} + e^{-\mu_B H/kT}}$$ ✓

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& $$N_\downarrow = N \frac{e^{-\mu_B H/kT}}{e^{\mu_B H/kT} + e^{-\mu_B H/kT}} .$$ ✓

Note that the denominator here may be written as $2 \cosh(\mu_B H/kT)$. $\longleftarrow$

(b) Net magnetization $M = N_\uparrow \cdot \mu_B + N_\downarrow \cdot (-\mu_B)$.

So $$M = N \frac{e^{\mu_B H/kT}}{2\cosh(\cdots)} \mu_B + N \frac{e^{-\mu_B H/kT}}{2\cosh(\cdots)} (-\mu_B)$$

$$= N\mu_B \frac{2\sinh(\cdots)}{2\cosh(\cdots)}$$

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$$= \underbrace{N\mu_B}_{\text{(the saturation value of } M.)} \tanh(\mu_B H/kT).$$ ✓

The net energy $U = N_\uparrow(-\mu_B H) + N_\downarrow(+\mu_B H)$

$$= -MH.$$

(3) So, $U = -N\mu_B H . \tanh(\mu_B H / kT)$.

In the Curie regime, $\mu_B H / kT << 1$, we get

(2) $$M \simeq N\mu_B . \frac{\mu_B H}{kT} = N . \frac{\mu_B^2 H}{kT},$$

which agrees with (17.19), with $J = \frac{1}{2}$ & $g = 2$.

(c) The total # of microstates $W = \dfrac{N!}{N_\uparrow ! \, N_\downarrow !}$, whence

(2)

$$S = k \ln W = k\left[\ln N! - \ln N_\uparrow ! - \ln N_\downarrow !\right]$$

$$\simeq k\left[(N \ln N - \cancel{N}) - (N_\uparrow \ln N_\uparrow - \cancel{N_\uparrow}) - (N_\downarrow \ln N_\downarrow - \cancel{N_\downarrow})\right]$$

Write this N as $(N_\uparrow + N_\downarrow)$, to get

$$S = k\left[N_\uparrow \ln \frac{N}{N_\uparrow} + N_\downarrow \ln \frac{N}{N_\downarrow}\right]$$; see Problem 15-2, part (a).

$$= kN\left[\frac{N_\uparrow}{N} \ln \frac{N}{N_\uparrow} + \frac{N_\downarrow}{N} \ln \frac{N}{N_\downarrow}\right].$$

We now substitute from part (a) of this problem, to get

$$S = Nk\left[\frac{e^{x}}{2\cosh x}\left\{ \ln(2\cosh x) - x \right\} + \frac{e^{-x}}{2\cosh x}\left\{ \ln(2\cosh x) + x \right\} \right] \qquad x = \frac{\mu_B H}{kT}$$

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$$= Nk\left[\ln(2\cosh x) - x\tanh x \right]. \checkmark \qquad (17.29)$$

extra comment

Note that, in this notation, $U = -NkT \cdot x\tanh x$. It follows that the Helmholtz free energy of the system is given by

$$F \equiv U - TS = -NkT\ln Z,$$

where $Z\,(= 2\cosh x)$ is the partition function of a single magnetic dipole.

(d) $$C_H = T\left(\frac{\partial S}{\partial T}\right)_H = T\,\frac{\partial S}{\partial x}\cdot\left(\frac{\partial x}{\partial T}\right)_H$$

$$= T.Nk\left[\cancel{\frac{2\sinh x}{2\cosh x}} - \cancel{\tanh x} - x.\operatorname{sech}^2 x \right]\cdot\frac{-\mu_B H}{kT^2}$$

$$= T.Nk\left[-\frac{\mu_B H}{kT}\operatorname{sech}^2\frac{\mu_B H}{kT} \right]\cdot\frac{-\mu_B H}{kT^2}$$

4

$$= Nk\left(\frac{\mu_B H}{kT}\right)^2 \operatorname{sech}^2\frac{\mu_B H}{kT}, \checkmark$$

which agrees with (17.27)

(e) In the Curie regime, $x << 1$, we have:

$$2\cosh x = 2\left(1 + \frac{x^2}{2} + \ldots\right),$$

so $$\ln(2\cosh x) \simeq \ln 2 + \frac{x^2}{2}.$$

At the same time, $x \tanh x \simeq x^2$.

So,

$$S \simeq Nk\left[\ln 2 + \frac{x^2}{2} - x^2\right]$$

3 $$= Nk\left[\ln 2 - \frac{x^2}{2}\right].$$

$$\longrightarrow \underline{Nk \ln 2} \text{ as } T \longrightarrow \infty.$$ [Interpret this result physically!]

From here, we get

$$C_H = T \frac{\partial S}{\partial x}\left(\frac{\partial x}{\partial T}\right)_H = TNk\left[-x\right] \cdot \frac{-\mu_B H}{kT^2}$$

2 $$= Nk \cdot \left(\mu_B H / kT\right)^2.$$

The same result follows readily from (17.27) as well.