1. Carter Problem 19-7.

(a) $$\varepsilon_F = \frac{h^2}{2m_e}\left(\frac{3N_e}{8\pi V}\right)^{2/3}, \text{ where } \frac{N_e}{V} = 3\times\frac{N_{Al}}{V} = 3\times\frac{\rho_{Al}}{m_{Al}}.$$

$$\therefore \varepsilon_F = \frac{(6.626\times10^{-34})^2}{2\times9.109\times10^{-31}}\left(\frac{3}{8\pi}\times3\times\frac{2.69\times10^3}{27\times1.661\times10^{-27}}\right)^{2/3}$$

(4) $$= 1.86\times10^{-18}\ \text{J} = \underline{11.6\ \text{eV}}.$$

(b) By (19.15), $$\frac{\mu(T)}{\mu(0)} \simeq 1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2.$$

Now, $$\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2 = \frac{\pi^2}{12}\left(\frac{1.381\times10^{-23}\times1000}{1.86\times10^{-18}}\right)^2$$

(2) $$= \underline{4.5\times10^{-5}},$$

clearly less than 0.01%, which is 10^{-4}.

(c) $$C_V = \frac{\pi^2}{2}N_e k.\left(\frac{kT}{\varepsilon_F}\right).$$

Since we are asked to compare its value at room temperature with $3R$, we are expected to consider <u>one kilomole</u> of aluminium. Hence

$$C_V = \frac{\pi^2}{2}\times(3\times6.022\times10^{26})\times(1.381\times10^{-23})^2\times300/1.86\times10^{-18}$$

(4) $$= \underline{274\ \text{J kmole}^{-1}\ \text{K}^{-1}}.$$

(2) The ratio $\frac{C_V}{3R} = \underline{0.011}$, which is a little over 1%.

2. $w = \frac{1}{\sqrt{\rho k_S}}$, where $\rho = \frac{mN}{V}$ & $k_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{S,N}$.

For an ideal Fermi gas at $T = 0\,K$,

$$U = \frac{3}{5} N \epsilon_F = \frac{3}{5} N \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3} \propto V^{-2/3}.$$

So, $$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N} = +\frac{2}{3}\frac{U}{V} = \frac{2}{5}\frac{N}{V}\epsilon_F \propto V^{-5/3}.$$

$$\therefore \left(\frac{\partial P}{\partial V}\right)_{S,N} = -\frac{5}{3}\frac{P}{V}$$ and hence

$$k_S = \frac{3}{5P} = \frac{3}{2}\frac{V}{N}\frac{1}{\epsilon_F}$$ and hence

$$\rho k_S = \frac{3m}{2\epsilon_F}.$$

(10) Thus, $$w = \sqrt{\frac{2\epsilon_F}{3m}} = \frac{1}{\sqrt{3}} v_F \quad \left(\because \epsilon_F = \frac{1}{2} m v_F^2\right).$$

3. For the given system,

$$U = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

(a) To determine F, we need S and to determine S, we need C_V.

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V = \left(\frac{\partial U}{\partial T} \right)_{\varepsilon_F} \quad \text{because } \varepsilon_F = f(N/V).$$

$$\therefore C_V = \frac{3}{5} N \varepsilon_F \cdot \frac{5\pi^2}{12} \cdot \frac{2k^2 T}{\varepsilon_F^2} + \cdots$$

$$= \frac{\pi^2}{2} Nk \left(\frac{kT}{\varepsilon_F} \right) + \cdots,$$

and hence

$$S = \int_0^T \frac{C_V \, dT}{T} = \frac{\pi^2}{2} Nk \left(\frac{kT}{\varepsilon_F} \right) + \cdots.$$

Thus

$$F = U - TS = \frac{3}{5} N \varepsilon_F \left[1 - \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

P can be determined from the relation $P = -(\partial F/\partial V)_T$ or from $P = -(\partial U/\partial V)_S$. In either case, we need to know how ε_F depends on V. With $\varepsilon_F \propto V^{-2/3}$, we get

8

for P & F

$$P = \frac{2}{3} \frac{U}{V} = \frac{2}{5} \frac{N}{V} \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

(b) Finally, $\mu = \frac{G}{N} = \frac{F + PV}{N}$

3

$$= \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

4. If one uses the <u>nonrelativistic</u> formula for ε_F, one gets

$$\varepsilon_F = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{2/3} = mc^2,$$

so that

$$n = \frac{8\pi}{3}\left(\frac{2m^2c^2}{h^2}\right)^{3/2} = \frac{16\sqrt{2}\,\pi}{3}\left(\frac{mc}{h}\right)^3$$

$$= 1.659 \times 10^{36}\ m^{-3}.$$

However, v_F (using <u>nonrelativistic</u> formula $\varepsilon_F = \frac{1}{2}mv_F^2$) turns out to be $\sqrt{2}\,c$, which is clearly wrong. So we must use <u>relativistic</u> formulae.

——— . ——— . ———

Using the relation

$$\varepsilon = c\sqrt{p^2 + m^2c^2} - mc^2,$$

we see that $\varepsilon_F = mc^2$ implies that $p_F = \sqrt{3}\,mc$.

However, p_F is given by the general formula

$$p_F = h\left(\frac{3n}{8\pi}\right)^{1/3}.$$

Equating this expression with $\sqrt{3}\,mc$, we get

(7) $$n = 8\sqrt{3}\pi\left(\frac{mc}{h}\right)^3 = 3.047 \times 10^{36}\ m^{-3}.$$

For v_F, we get

$$v_F = \frac{p_F}{m_{rel.}} = \frac{p_F c^2}{E_{total}} = \frac{(\sqrt{3}\,mc)\,c^2}{2mc^2} = \frac{\sqrt{3}}{2}c$$

$$= 0.886\,c$$

(3) $$= 2.596 \times 10^8\ m/s.$$

5.

(a) With the given approximation, the ground-state energy of the electron gas in a white dwarf star turns out to be

$$U_0 = \int_0^{p_F} 2 \cdot \frac{V \cdot 4\pi p^2 dp}{h^3} \left(cp - m_e c^2 + \frac{m_e^2 c^3}{2p} \right)$$

$$= \frac{8\pi V}{h^3} \left(c \frac{p_F^4}{4} - m_e c^2 \frac{p_F^3}{3} + \frac{m_e^2 c^3}{2} \frac{p_F^2}{2} \right), \qquad (1)$$

where

$$p_F = h \left(\frac{3N}{8\pi V} \right)^{1/3}. \qquad (2)$$

The middle term in (1) is simply the rest energy of the gas; being a constant, it does not play any role in determining the equilibrium configuration of the star. The other two terms are

$$\frac{2\pi h c}{V^{1/3}} \left(\frac{3N}{8\pi} \right)^{4/3} + \frac{2\pi m_e^2 c^3}{h} V^{1/3} \left(\frac{3N}{8\pi} \right)^{2/3},$$

which may be written as

$$a \frac{M^{4/3}}{R} + b M^{2/3} R, \qquad (3)$$

where

5

$$a \sim \frac{hc}{m_{He}^{4/3}} \quad \text{and} \quad b \sim \frac{m_e^2 c^3}{h \cdot m_{He}^{2/3}}. \qquad (4)$$

The gravitational energy of the star is given by

$$E_g \sim -\alpha \frac{GM^2}{R} \qquad \alpha = O(1). \tag{5}$$

Adding (3) and 5, and minimizing the sum with respect to R, the condition for equilibrium turns out to be

$$-a\frac{M^{4/3}}{R^2} + b\,M^{2/3} + \alpha\frac{GM^2}{R^2} = 0, \tag{6}$$

which yields the desired result

$$\begin{aligned} R &= \sqrt{\frac{a}{b}}\, M^{1/3} \left[1 - \frac{\alpha G}{a} M^{2/3}\right]^{1/2} \\ &\sim \frac{h}{m_e c}\left(\frac{M}{m_{He}}\right)^{1/3}\left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2}, \end{aligned} \tag{7}$$

where

$$M_0 = \left(\frac{a}{\alpha G}\right)^{3/2} \sim \frac{(hc/G)^{3/2}}{m_{He}^2}. \tag{8}$$