Physics 140B: Homework 2 Solutions

1. a) By equation (11.28) of the text, the Maxwell velocity distribution is

$$N(v)dv = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-(\frac{1}{2}mv^2)/kT} \cdot 4\pi v^2 dv$$

Using $\varepsilon = \frac{1}{2}mv^2$ we can rephrase this in terms of the energy.

$$N(\varepsilon)d\varepsilon = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\varepsilon/kT} 4\pi \underbrace{\left(\frac{2\varepsilon}{m}\right)}_{v^2} \cdot \underbrace{\left(\frac{2}{m}\right)^{1/2}\frac{1}{2}\varepsilon^{-1/2}d\varepsilon}_{dv}$$

$$= N\frac{2}{\sqrt{\pi}(kT)^{3/2}}e^{-\varepsilon/kT}\varepsilon^{1/2}d\varepsilon$$

Now, we use $N = N_A$, $T = 273.15K$, $\varepsilon = \bar{\varepsilon} = \frac{3}{2}kT$, and $d\varepsilon = 10^{-22}J$ we get

$$\boxed{N(\varepsilon)d\varepsilon = 4.9 \times 10^{24}}$$

b) Using equation (12.25) the number of "single-particle energy states" in a small interval is given by

$$g(\varepsilon)d\varepsilon = \frac{4\sqrt{2}\pi V}{h^3}m^{3/2}\varepsilon^{1/2}d\varepsilon$$

Making the same substitutions as above we get

$$\boxed{g(\varepsilon)d\varepsilon = 5.6 \times 10^{30}}$$

c) Using the above expressions we can compute the ratio

$$\frac{N(\varepsilon)d\varepsilon}{g(\varepsilon)d\varepsilon} = \underbrace{\frac{Nh^3}{V(2\pi mkT)^{3/2}}}_{\text{Note that this is nothing but } e^{\beta\mu}} \underbrace{e^{-3/2}}_{\text{And this is } e^{-\beta\varepsilon}} = \boxed{8.8 \times 10^{-7}}$$

Also, note that $\frac{N(\varepsilon)}{g(\varepsilon)} \ll 1$, which justifies the use of Maxwell-Boltzmann statistics!

2. As shown in class the expectation value of the number of photons in a radiation cavity is [1]

$$\bar{N} = 2.404 \cdot 8\pi V \left(\frac{kT}{hc}\right)^3$$

Thus, plugging in the average temperature of the CMB, $T = 2.7K$ along with the constants we get

$$\boxed{\bar{N} = 1.67 \times 10^{87}}$$

3. Given [2]

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P$$

In the context of blackbody radiation we may write

$$U = V \cdot u(T) \quad , \quad P = \frac{1}{3}u(T)$$

and plug into the above relation to get

$$u = T \cdot \frac{1}{3}\frac{du}{dT} - \frac{1}{3}u \Rightarrow \boxed{T\frac{du}{dT} = 4u}$$

which is the desired differential equation for $u(T)$, (note that it is key that u is only a function of temperature so we can make the partial derivative a total derivative). We can now solve for the explicit T dependence

$$\frac{du}{u} = 4\frac{dT}{T} \Rightarrow \ln u = 4\ln T + K \Rightarrow \boxed{u = cT^4}$$

[1] Note, this expression simply comes from $\bar{N} = \int_0^\infty N(\nu)d\nu = \int_0^\infty \frac{g(\nu)d\nu}{e^{h\nu/kT}-1} = 8\pi V\left(\frac{kT}{hc}\right)^3\int_0^\infty \frac{x^2dx}{e^x-1}$ and the dimensionless integral can be evaluated numerically to give 2.404.

[2] For reference, see equation (6.26)

4. Start from equation (18.44)

$$U = AT^{5/2}, \quad \text{where } A = \underbrace{0.77NkT_B^{-3/2}}_{\text{a function of N\&V}}$$

Then,

$$C_v = \left(\frac{\partial U}{\partial T}\right)_{N,V} = \frac{5}{2}AT^{3/2} = \boxed{\frac{5}{2}\frac{U}{T}}$$

$$S = \int_0^T \frac{C_v dT}{T} = \int_0^T \frac{5}{2}AT^{1/2}dT = \frac{5}{2}A\left(\frac{2}{3}T^{3/2}\right) = \frac{5}{3}AT^{3/2} = \boxed{\frac{5}{3}\frac{U}{T}}$$

$$F = U - TS = AT^{5/2} - \frac{5}{3}AT^{5/2} = -\frac{2}{3}AT^{5/2} = \boxed{-\frac{2}{3}U}$$

$$PV = G - F = N\mu - F = 0 - \left(-\frac{2}{3}U\right) = \frac{2}{3}U \Rightarrow \boxed{P = \frac{2}{3}\frac{U}{V}}$$

5. The average number of particles in a given energy state for a Bose-Einstein gas is given by

$$\bar{N}_\varepsilon = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1} \quad (\varepsilon = Ap^s)$$

In the region of Bose-Einstein condensation, μ is essentially zero [3] Thus,

$$N_{exc} = \int_0^\infty \bar{N}_\varepsilon g(\varepsilon) d\varepsilon$$

where we can derive the density of states $g(\varepsilon)$ from the "phase-space" expression:

$$\frac{V \cdot 4\pi p^2 dp}{h^3} = \frac{4\pi V}{h^3}\left(\frac{\varepsilon}{A}\right)^{2/s}\frac{1}{s}\left(\frac{\varepsilon}{A}\right)^{1/s-1} d\varepsilon \sim V\varepsilon^{3/s-1}d\varepsilon$$

Thus,

$$\begin{aligned} N_{exc} &= \text{const} \cdot\ V\int_0^\infty \frac{\varepsilon^{3/s-1}d\varepsilon}{e^{\varepsilon/kT} - 1} \quad \left[\text{set } \frac{\varepsilon}{kT} = x\right] \\ &= \text{const} \cdot\ V(kT)^{3/s} \propto T^{3/s} \end{aligned}$$

a) T_B is determined by the condition $N_{exc} = N$, it follows that $\boxed{T_B \propto \left(\frac{N}{V}\right)^{s/3}}$.

[3] This is because at temperatures near zero, $N_0 \approx N$ and so $\varepsilon \approx 0$. This implies that $N \approx (e^{-\mu/kT} - 1)^{-1}$ and thus $-\mu/kT \approx \ln\left(1 + \frac{1}{N}\right) \approx \frac{1}{N}$, so for a large collection of particles the chemical potential is essentially zero in the region near Bose-Einstein condensation.

b) Since $N_{exc} \propto T^{3/s}$ we get

$$\frac{N_{exc}}{N} = \left(\frac{T}{T_B}\right)^{3/s} \therefore \boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_B}\right)^{3/s}}.$$

c) it is now straightforward to show that

$$U = \int_0^\infty \varepsilon \bar{N}(\varepsilon) g(\varepsilon) d\varepsilon \sim T^{3/s+1}$$

Hence, $\boxed{C_v \sim T^{3/s}}$ and $S = \int_0^T \frac{C_v dT}{T} \Rightarrow \boxed{S \sim T^{3/s}}$.