Physics 140B: Homework 1 Solutions

1. The total number of diatomic molecules $= 1,000$. The temperature of the system $= \frac{1}{2}\theta_{vib}$.

   a) By equation (15.6),

$$\bar{N_j} = Ne^{-2j}(1-e^{-2})$$

   Thus we get

$$\begin{aligned}
\bar{N_0} &= 1000(1-e^{-2}) \approx 865 \quad (j=0) \\
\bar{N_1} &= 1000e^{-2}(1-e^{-2}) \approx 117 \quad (j=1) \\
\bar{N_2} &= 1000e^{-4}(1-e^{-2}) \approx 16 \quad (j=2)
\end{aligned}$$

   b) By equation (15.8)

$$\frac{U}{N} = k\theta_{vib}\left(\frac{1}{2} + \frac{1}{e^2-1}\right) \approx 0.6565k\theta_{vib}$$

2.

$$p_j = \frac{g_j e^{-\varepsilon_j/kT}}{Z} = \frac{(2j+1)e^{-j(j+1)\hbar^2/2IkT}}{\sum_j (2j+1)e^{-j(j+1)\hbar^2/2IkT}} \tag{1}$$

For $T \gg \theta_{rot}\left(\equiv \frac{\hbar^2}{2Ik}\right)$, the eigenvalue ε_j (and the quantum number j) may be treated as continuous variables, so that

$$d\varepsilon = \frac{\hbar^2}{2I}(2j+1)dj \tag{2}$$

It follows from Equations (1) and (2) that

$$p(\varepsilon)d\varepsilon = \frac{e^{-\varepsilon/kT}d\varepsilon/k\theta_{rot}}{\int_0^\infty e^{-\varepsilon/kT}d\varepsilon/k\theta_{rot}} = \frac{1}{kT}e^{-\varepsilon/kT}d\varepsilon$$

The desired fraction then turns out to be

$$\int_{kT}^{\infty} \frac{1}{kT} e^{-\varepsilon/kT} d\varepsilon = -e^{-\varepsilon/kT}\Big|_{kT}^{\infty} = e^{-1} \approx 0.368$$

3. According to the *Einstein* model, (Equation 16.2)

$$C_v = 3Nk \left(\frac{\theta_E}{T}\right)^2 \frac{e^{\theta_E/T}}{(e^{\theta_E/T} - 1)^2}$$

With $N = N_A$, $T = 150K$ and $\theta_E = 1450K$, we get

$$C_v = 147.7J \,\text{kilomole}^{-1}\, K^{-1}$$

According to the *Debye* model (Equation 16.19), since $T < 0.1\theta_D$,

$$C_v = \frac{12\pi^4}{5} Nk \left(\frac{T}{\theta_D}\right)^3$$

With $N = N_A$, $T = 150K$ and $\theta_D = 1860K$, we get

$$C_v = 1019J \,\text{kilomole}^{-1}\, K^{-1}$$

4. a) Wien's Displacement Law says that

$$\lambda_{max} \cdot T = 2.90 \times 10^{-3} m\ K$$

With, $\lambda_{max} = 480 \times 10^{-9} m$, we obtain $T = 6040K$. Note that λ_{max} falls well within the visible part of the spectrum!

- Next, the radiative flux (per unit surface area per unit time) $= \sigma T^4$ (Stefan-Boltzmann Law).

$\therefore$ Total Radiative Power emitted by the Sun, L is given by

$$\begin{aligned} L &= \sigma T^4 \cdot A_{Sun} \\ &= \sigma T^4 (4\pi R_{sun}^2) \\ &= (5.67 \times 10^{-8} W m^{-2} K^{-4}) \times (6040K)^4 \times 4\pi (7 \times 10^8 m)^2 \approx 4.65 \times 10^{26} W \end{aligned}$$

5. Solar radiative flux observed on the surface of the Earth, Φ is given by

$$\Phi = \frac{L}{4\pi D^2} = \frac{4.65 \times 10^{26} W}{4\pi (1.5 \times 10^{11} m)^2} \approx 1645 W m^{-2}$$

where D is the distance between the Sun and the Earth.

The total power received by the Earth would be Φ multiplied by the total *normal area* presented by the Earth towards the solar rays, which is πR^2_{earth}, *i.e.* the surface area of the "equatorial disk".

$$P_{abs} = \Phi \cdot \pi R^2_{earth} \tag{3}$$

Now, in the steady state, the Earth emits the same power from *all over its surface*, whose area is $4\pi R^2_{earth}$. If T is the steady-state temperature of the Earth, then this radiated power would be $P_{rad} = \sigma T^4 \cdot 4\pi R^2_{earth}$. Thus, equating the two powers, we get

$$T = \left(\frac{1645}{5.67 \times 10^{-8} \cdot 4} \right)^{1/4} \approx 292K$$

which is pretty darn close!

6. Problem 18-4 of the text

   a) Differentiate expression (18.5) with respect to ν and set the result equal to zero, one gets

   $$\nu_{max} = \frac{kT}{h} \cdot x$$

   where x is the solution of the equation

   $$e^x \left(1 - \frac{x}{3} \right) = 1$$

   A numerical solution gives: $x \approx 2.8214 \cdots$. Hence $\nu_{max} \approx \frac{kT}{h} \cdot 2.8214$.

   The value of λ that maximized expression (18.6) instead is $\frac{hc}{4.9651 kT}$, which corresponds to a frequency $\nu = \frac{kT}{h} \cdot 4.9651$.

   Thus, our ν_{max} is smaller than this ν by a factor of 1.76.

   b) In the case of the Cosmic Microwave Background

   $$\nu_{max} = \frac{1.381 \times 10^{-23} J K^{-1} \cdot 2.7K}{6.626 \times 10^{-34} J sec} \times 2.8214 \approx 1.59 \times 10^{11} Hz$$

   The corresponding wavelength is 1.89 mm, which falls in the "microwave" range!