Handout 2

We'll evaluate here the specific heat C_V of an ideal Fermi gas at temperatures such that $kT \ll \varepsilon_F$.

We start with the obvious expressions

$$N = \int_0^\infty \frac{C \cdot \varepsilon^{1/2}\, d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1} \quad \& \quad U = \int_0^\infty \frac{C \cdot \varepsilon^{3/2}\, d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}, \qquad (1,2)$$

where

$$C = \frac{8\sqrt{2}\,\pi V m^{3/2}}{h^3}. \qquad (3)$$

With $\frac{\varepsilon}{kT} = x$ & $\frac{\mu}{kT} = \xi$, eqns. (1) & (2) become

$$N = C \cdot (kT)^{3/2} \int_0^\infty \frac{x^{1/2}\, dx}{e^{x-\xi}+1}, \text{ and} \qquad (4)$$

$$U = C \cdot (kT)^{5/2} \int_0^\infty \frac{x^{3/2}\, dx}{e^{x-\xi}+1}. \qquad (5)$$

So, let's consider the integral

$$I = \int_0^\infty \frac{x^{\nu-1}\, dx}{e^{x-\xi}+1}, \text{ where } \xi \gg 1. \qquad (6)$$

In view of the fact that, at low temperatures, N_ε (for $\varepsilon < \mu$) is only a little bit less than 1 and (for $\varepsilon > \mu$) only a little bit more than 0, we may write

$$I = \int_0^{\xi} \left[1 - \frac{1}{e^{\xi - x} + 1}\right] x^{\nu-1}\,dx + \int_{\xi}^{\infty} \frac{1}{e^{x-\xi}+1} x^{\nu-1}\,dx$$

$$= \frac{\xi^\nu}{\nu} - \int_0^{\xi} \frac{x^{\nu-1}\,dx}{e^{\xi-x}+1} + \int_{\xi}^{\infty} \frac{x^{\nu-1}\,dx}{e^{x-\xi}+1}.$$

In the first integral here, we set $x = \xi - \eta_1$ and in the second, $x = \xi + \eta_2$, with the result that

$$I = \frac{\xi^\nu}{\nu} - \int_{\xi}^{0} \frac{(\xi - \eta_1)^{\nu-1}(-d\eta_1)}{e^{\eta_1}+1} + \int_0^{\infty} \frac{(\xi+\eta_2)^{\nu-1}\,d\eta_2}{e^{\eta_2}+1}.$$

Since $\xi \gg 1$, we may in the first integral replace the lower limit by ∞, so that

$$I = \frac{\xi^\nu}{\nu} + \int_0^{\infty} \frac{-(\xi-\eta)^{\nu-1} + (\xi+\eta)^{\nu-1}}{e^\eta + 1}\,d\eta.$$

Since $\xi \gg 1$ and η is effectively $O(1)$, we may invoke binomial expansion and write

$$I \simeq \frac{\xi^\nu}{\nu} + \int_0^{\infty} \frac{2(\nu-1)\,\xi^{\nu-2}\,\eta}{e^\eta+1}\,d\eta. \qquad (7)$$

Now we look at the integral $\int_0^\infty \frac{\eta\, d\eta}{e^\eta + 1}$, which may be written as

$$\int_0^\infty \left(e^{-\eta} - e^{-2\eta} + e^{-3\eta} - e^{-4\eta} + \dots\right)\eta\, d\eta$$

$$= \Gamma(2)\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$= \Gamma(2)\left[\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right)\right]$$

$$= \Gamma(2)\left[\zeta(2) - \frac{2}{2^2}\zeta(2)\right]$$

$$= \Gamma(2)\,\zeta(2) \cdot \frac{1}{2}$$

$$= 1 \cdot \frac{\pi^2}{6} \cdot \frac{1}{2} = \frac{\pi^2}{12}.$$

Hence, our original integral

$$I \simeq \frac{\xi^\nu}{\nu} + \frac{\pi^2}{6}(\nu - 1)\,\xi^{\nu-2}$$

$$= \frac{\xi^\nu}{\nu}\left[1 + \frac{\pi^2}{6}\nu(\nu-1)\,\xi^{-2}\right]. \qquad (8)$$

Equations (4) & (5) now become

$$N \simeq C \cdot \frac{2}{3}\mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] \qquad (9)$$

and

$$U \simeq C \cdot \frac{2}{5}\mu^{5/2}\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]. \qquad (10)$$

Now, at $T=0$, we had

$$N = C \cdot \frac{2}{3}\mu_0^{3/2} \qquad (\mu_0 \equiv \varepsilon_F);$$

therefore, at finite T such that $kT \ll \varepsilon_F$,

$$\mu\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right]^{2/3} \simeq \mu_0, \quad \text{i.e.}$$

$$\mu \simeq \mu_0\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right], \qquad (11)$$

which is Carter's eqn. (19.15).

Now, by eqns. (9) & (10),

$$\frac{U}{N} \simeq \frac{3}{5}\mu\left[1+\frac{\pi^2}{2}\left(\frac{kT}{\mu}\right)^2\right]$$

which, by virtue of (11), becomes

$$\frac{U}{N} \simeq \frac{3}{5}\mu_0\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right], \text{ so that}$$

$$U_0 = \frac{3}{5}N\mu_0 \;\&\; U_{\text{thermal}} \simeq \frac{\pi^2}{4}N\frac{k^2T^2}{\mu_0}.$$

It follows that $C_V \simeq \frac{\pi^2}{2}Nk \cdot \frac{kT}{\mu_0}$ [Sommerfeld, 1928].