Handout 1

Consider the integral:

$$I = \int_0^\infty \frac{x^{\nu-1}}{e^x - 1}\,\mathrm{d}x \tag{1}$$

We can write:

$$\begin{aligned} \frac{1}{e^x - 1} &= \frac{e^{-x}}{1 - e^{-x}} = e^{-x}(1 - e^{-x})^{-1} \\ &= e^{-x} + e^{-2x} + e^{-3x} + \ldots = \sum_{j=1}^{\infty} e^{-jx} \end{aligned} \tag{2}$$

The first term in the sum represents the <u>classical limit</u>.
Thus (1) becomes:

$$I = \sum_{j=1}^{\infty} \int_0^\infty e^{-jx} x^{\nu-1} \mathrm{d}x = \sum_{j=1}^{\infty} \frac{\Gamma(\nu)}{j^\nu} = \Gamma(\nu)\zeta(\nu) \tag{3}$$

where $\zeta(\nu)$ is the <u>Riemann Zeta Function</u> of order ν

$$\zeta(\nu) = \sum_{j=1}^{\infty} \frac{1}{j^\nu} = 1 + \frac{1}{2^\nu} + \frac{1}{3^\nu} + \ldots \qquad (\nu > 1) \tag{4}$$

The 1 is the <u>classical</u> term.
The <u>numerical factors</u> we encountered in the class were:

$$\begin{aligned} \zeta\left(\frac{3}{2}\right) &= 2.612\ldots \\ \zeta\left(\frac{5}{2}\right) &= 1.341\ldots \\ \zeta(3) &= 1.202\ldots \end{aligned} \tag{5}$$

In the study of "black-body radiation," we came across the integral:

$$I = \int_0^\infty \frac{x^3}{e^x - 1}\mathrm{d}x \tag{6}$$

From (3), this integral is equal to $\Gamma(4)\zeta(4)$.
Now, it so happens that $\zeta(\nu)$, when ν is an <u>even integer</u>, can be expressed

in a closed form; for instance,

$$\begin{aligned}\zeta(2) &= \frac{\pi^2}{6} = 1.645\ldots \\ \zeta(4) &= \frac{\pi^4}{90} = 1.082\ldots \\ \zeta(6) &= \frac{\pi^6}{945} = 1.017\ldots,\end{aligned} \tag{7}$$

From this we see that the integral in (6) is precisely equal to $6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15}$, as was quoted in the class.