Physics 140B, Winter 2010
Midterm exam
Feb 16, 11:00 a.m. – 12:20 p.m.

Notes: (i) Attempt as many questions as you can.
(ii) All questions may not be of equal value!

1. (a) Show that the Debye temperature of a solid is given by the formula

$$\Theta_D = (hc/k)\,(3N/4\pi V)^{1/3},$$

where the various symbols have their usual meanings; note that the symbol c here stands for the speed of sound waves --- both longitudinal and transverse.

(b) For copper (atomic weight 63.6), the mass density ρ is 8.95×10^3 kg.m^{-3} while the speed of sound c is 2.24×10^3 m.s^{-1}. Using these data, determine Θ_D for copper.

2. Consider a gas of N identical atoms trapped in a magnetic field such that they behave like indistinguishable simple harmonic oscillators obeying Bose-Einstein statistics. The energy of any one of those atoms is given by the expression

$$\varepsilon = p^2/2m + (1/2)Kr^2,$$

so that they all vibrate with a common frequency

$$\nu = (1/2\pi)\,(K/m)^{1/2}.$$

(a) Show that, at low temperatures, this gas undergoes the phenomenon of Bose-Einstein condensation, with

$$T_B \sim (h\nu/k)\,N^{1/3}.$$

(b) Next, determine the manner in which the condensate fraction N_0/N and the specific heat C_v of the gas vary with T --- when $T<T_B$.

3. The Helmholtz free energy of a non-relativistic Fermi gas at low temperatures is given by the expression

$$F = (3/5)N\,\varepsilon_F\,[\,1 - (5\pi^2/12)\,(kT/\varepsilon_F)^2 + \text{-------}\,],$$

where ε_F is the Fermi energy of the gas; note that $\varepsilon_F \propto (N/V)^{2/3}$.

(a) Using this expression for F, derive the corresponding expression for the chemical potential μ of the gas.

4. Consider an electron gas with particle density n. Determine the *numerical* value of n for which the Fermi momentum p_F of the gas is equal to mc, where m is the rest-mass of the electron and c the speed of light.

What is the corresponding value of the Fermi energy ε_F (in units of mc^2) and the Fermi velocity v_F (in units of c)?

5. Using Maxwell's relations, show that for a magnetic system

$$(\,\partial C_H / \partial H\,)_T = T(\,\partial^2 M / \partial T^2\,)_H\,.$$

Apply this result to a paramagnetic material *in the Curie regime* and show that, in this case,

$$C_H = C\,H^2/T^2 + f\,(T),$$

where C is the Curie constant and f (T) an unknown function of T .

1. (a) $\Theta_D = h\nu_{max}/k$, where ν_{max} is determined by the condition

$$\int_0^{\nu_{max}} 4\pi V \left(\frac{1}{c_l^3} + \frac{2}{c_t^3}\right) \nu^2 \, d\nu = 3N.$$

Dropping the distinction between c_l & c_t, we get

$$4\pi V \frac{3}{c^3} \frac{\nu_{max}^3}{3} = 3N. \qquad \therefore \nu_{max} = c\left(\frac{3N}{4\pi V}\right)^{1/3}.$$

It follows that

4 $$\Theta_D = \frac{hc}{k}\left(\frac{3N}{4\pi V}\right)^{1/3}. \checkmark$$

(b) Since $\rho = N m_{cu}/V$, we get

$$\Theta_D = \frac{hc}{k}\left(\frac{3\rho}{4\pi m_{cu}}\right)^{1/3}$$

$$= \frac{6.626\times10^{-34}\,\mathrm{J\,s}\cdot 2.24\times10^{3}\,\mathrm{m\,s^{-1}}}{1.381\times10^{-23}\,\mathrm{J\,K^{-1}}}\left(\frac{3\times8.95\times10^{3}\,\mathrm{kg\,m^{-3}}}{4\pi\times63.6\times1.661\times10^{-27}\,\mathrm{kg}}\right)^{1/3}$$

4 $$= 293\ \mathrm{K}. \checkmark$$

2. (a) In the region of BEC, $\mu = \varepsilon_{min} \simeq 0$. So, the expectation value of the total # of atoms in all the excited states is

$$N_{exc} = \sum_{\varepsilon > \varepsilon_{min}} \bar{N}_\varepsilon \approx \int_0^\infty \frac{g(\varepsilon)\, d\varepsilon}{e^{\varepsilon/kT} - 1}$$

$$\approx \int_0^\infty \int_0^\infty \frac{4\pi p^2 dp \cdot 4\pi r^2 dr / h^3}{e^{p^2/2mkT + Kr^2/2kT} - 1}.$$

Set $p = \sqrt{2mkT}\, u$ & $r = \sqrt{2kT/K}\, v$, to get

$$N_{exc} = 128\pi^2 (m/K)^{3/2} (kT/h)^3 \cdot I,$$

where
$$I = \int_0^\infty \int_0^\infty \frac{u^2 v^2\, du\, dv}{e^{u^2 + v^2} - 1}, \quad \underset{\sim\sim\sim}{\text{a numerical constant.}}$$

It follows that

$$N_{exc} = \text{const.} (kT/h\nu)^3.$$

Now, T_B is determined by the condition: $N_{exc} = N$, with the result that

6 $$T_B = \text{const.} (h\nu/k) \cdot N^{1/3}. \checkmark$$

(b) For $T < T_B$, $\underset{\sim\sim\sim}{N_{exc} \propto T^3}$ and, at $T = T_B$, $N_{exc} = N$. It follows that, for $T < T_B$,

(3) $$N_{exc}/N = (T/T_B)^3$$

and, hence,

$$N_0/N = 1-(T/T_B)^3.$$ ✓

As for C_V, we note that the <u>mean</u> thermal energy carried by an atom in an excited state is $O(kT)$. The total thermal energy of the

(3) gas is, therefore, $\propto T^4$ and, hence, its $C_V \propto T^3$. ✓

3. Since $dF = -S\,dT - P\,dV + \mu\,dN$, therefore $\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$.

The given expression for F is

$$F = \frac{3}{5} N \varepsilon_F - \frac{\pi^2}{4} N \frac{k^2 T^2}{\varepsilon_F} + \cdots$$

Since $\varepsilon_F \propto (N/V)^{2/3}$, the first term of F is $\propto N^{5/3}$ and the second term is $\propto N^{1/3}$; other factors are functions of T & V only, so they remain unaffected by differentiation w.r.t. N. We thus get

$$\mu = \frac{5}{3N} \cdot \frac{3}{5} N \varepsilon_F - \frac{1}{3N} \cdot \frac{\pi^2}{4} N \frac{k^2 T^2}{\varepsilon_F} + \cdots$$

8

$$= \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right] .$$

4. The Fermi momentum p_F is given by the condition

$$\int_0^{p_F} 2 \cdot \frac{V\, 4\pi p^2\, dp}{h^3} = N, \quad \text{i.e.} \quad p_F = h\left(\frac{3N}{8\pi V}\right)^{1/3}.$$

Equating p_F with mc, we get

(4) $$n = \frac{N}{V} = \frac{8\pi}{3}\left(\frac{mc}{h}\right)^3 = 5.84 \times 10^{35}\ m^{-3}.$$

The corresponding values of ε_F/mc^2 and v_F/c are given by

(2) $$\varepsilon_F = c\sqrt{p_F^2 + m^2c^2} - mc^2 = (\sqrt{2}-1)\,mc^2, \quad \therefore \quad \frac{\varepsilon_F}{mc^2} = \sqrt{2}-1, \text{ and}$$

(2) $$v_F = \frac{p_F}{m_{rel}} = \frac{p_F c^2}{\varepsilon_{total}} = \frac{mc \cdot c^2}{\sqrt{2}\, mc^2} = \frac{c}{\sqrt{2}}, \quad \therefore \quad \frac{v_F}{c} = \frac{1}{\sqrt{2}}.$$

5. $$\left(\frac{\partial C_H}{\partial H}\right)_T = \left[\frac{\partial}{\partial H}\left\{T\left(\frac{\partial S}{\partial T}\right)_H\right\}\right]_T$$

$$= T\,\frac{\partial^2 S}{\partial H\,\partial T} = T\,\frac{\partial^2 S}{\partial T\,\partial H}.$$

Using one of Maxwell's relations, we get

(4) $$T\left[\frac{\partial}{\partial T}\left(\frac{\partial M}{\partial T}\right)_H\right]_H = T\left(\frac{\partial^2 M}{\partial T^2}\right)_H.$$ ✓

For a paramagnetic material in the Curie regime,

$M = \dfrac{CH}{T}$. Therefore, $\left(\dfrac{\partial^2 M}{\partial T^2}\right)_H = \dfrac{2CH}{T^3}$.

It follows that

$$\left(\frac{\partial C_H}{\partial H}\right)_T = \frac{2CH}{T^2}.$$

Integrating over H, keeping T const., we get

(4) $$C_H = \frac{CH^2}{T^2} + \text{a const. independent of } H,$$ ✓

but possibly a function of T.

Hence the result.