

6-5. (a) $\Psi(x,t) = A \sin(kx - \omega t)$

$$\frac{\partial \Psi}{\partial t} = -\omega A \cos(kx - \omega t)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = -i\hbar \omega A \cos(kx - \omega t)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -k^2 A \sin(kx - \omega t)$$

$$\frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = \frac{-\hbar^2 k^2 A}{2m} \sin(kx - \omega t) \neq i\hbar \frac{\partial \Psi}{\partial t}$$

(b) $\Psi(x,t) = A \cos(kx - \omega t) + iA \sin(kx - \omega t)$

$$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar \omega A \sin(kx - \omega t) - i^2 \hbar \omega A \cos(kx - \omega t)$$

$$= \hbar \omega A \cos(kx - \omega t) + i\hbar \omega A \sin(kx - \omega t)$$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = \frac{\hbar^2 k^2 A}{2m} \cos(kx - \omega t) + \frac{\hbar^2 i k^2 A}{2m} \sin(kx - \omega t)$$

$$= \frac{\hbar^2 k^2}{2m} [A \cos(kx - \omega t) + iA \sin(kx - \omega t)]$$

$$= i\hbar \frac{\partial \Psi}{\partial t} \quad \text{if } \frac{\hbar^2 k^2}{2m} = \hbar \omega \text{ it does. (Equation 6-5 with } V=0)$$

6-9. (a) The ground state of an infinite well is $E_1 = h^2 / 8mL^2 = (hc)^2 / 8mc^2 L^2$

$$\text{For } m = m_p, L = 0.1 \text{ nm: } E_1 = \frac{(1240 \text{ MeV} \cdot \text{fm})^2}{8(938.3 \times 10^6 \text{ eV})(0.1 \text{ nm})^2} = 0.021 \text{ eV}$$

(b) For $m = m_p, L = 1 \text{ fm: } E_1 = \frac{(1240 \text{ MeV} \cdot \text{fm})^2}{8(938.3 \times 10^6 \text{ eV})(1 \text{ fm})^2} = 205 \text{ MeV}$

6-10. The ground state wave function is ($n = 1$) $\psi_1(x) = \sqrt{2/L} \sin(\pi x / L)$ (Equation 6-32)

The probability of finding the particle in Δx is approximately:

$$P(x)\Delta x = \frac{2}{L} \sin^2\left(\frac{\pi x}{L}\right) \Delta x = \frac{2\Delta x}{L} \sin^2\left(\frac{\pi x}{L}\right)$$

(a) For

$$x = \frac{L}{2} \text{ and } \Delta x = 0.002L, P(x)\Delta x = \frac{2(0.002L)}{L} \sin^2\left(\frac{\pi L}{2L}\right) = 0.004 \sin^2 \frac{\pi}{2} = 0.004$$

$$(b) \text{ For } x = \frac{2L}{3} \text{ and } P(x)\Delta x = \frac{2(0.002L)}{L} \sin^2\left(\frac{2\pi L}{3L}\right) = 0.004 \sin^2 \frac{2\pi}{3} = 0.0030$$

$$(c) \text{ For } x = L \text{ and } P(x)\Delta x = 0.004 \sin^2 \pi = 0$$

6-11. The second excited state wave function is ($n = 3$) $\psi_3(x) = \sqrt{2/L} \sin(3\pi x/L)$

(Equation 6-32). The probability of finding the particle in Δx is approximately:

$$P(x)\Delta x = \frac{2}{L} \sin^2\left(\frac{3\pi x}{L}\right) \Delta x$$

(a) For

$$x = \frac{L}{2} \text{ and } \Delta x = 0.002L, P(x)\Delta x = \frac{2(0.002L)}{L} \sin^2\left(\frac{3\pi L}{2L}\right) = 0.004 \sin^2 \frac{3\pi}{2} = 0.004$$

$$(b) \text{ For } x = \frac{2L}{3} \text{ and } P(x)\Delta x = 0.004 \sin^2\left(\frac{6\pi L}{3L}\right) = 0.004 \sin^2 2\pi = 0$$

$$(c) \text{ For } x = L \text{ and } P(x)\Delta x = 0.004 \sin^2\left(\frac{3\pi L}{L}\right) = 0.004 \sin^2 3\pi = 0$$

$$6-16. E_n = \frac{h^2 n^2}{8mL^2} \text{ and } \Delta E_n = E_{n+1} - E_n = \frac{h^2}{8mL^2} (n^2 + 2n + 1)$$

$$\text{or, } \Delta E_n = (2n + 1) \frac{h^2}{8mL^2} = \frac{hc}{\lambda}$$

$$\text{so, } L = \left(\frac{\lambda h}{8m\lambda}\right)^{1/2} = \left(\frac{\lambda hc}{8mc^2}\right)^{1/2} = \left(\frac{(694.3\text{nm})(1240\text{eV}\cdot\text{nm})}{8(0.511 \times 10^6 \text{eV})}\right)^{1/2} = 0.459\text{nm}$$

$$6-20. E_1 = \frac{h^2}{8mL^2} = \frac{(hc)^2}{8(mc^2)L^2}$$

$$(a) \text{ For an electron: } E_1 = \frac{(1240 \text{ MeV} \cdot \text{fm})^2}{8(0.511 \text{ MeV})(10 \text{ fm})^2} = 3.76 \times 10^3 \text{ MeV}$$

$$(b) \text{ For a proton: } E_1 = \frac{(1240 \text{ MeV} \cdot \text{fm})^2}{8(938.3 \text{ MeV})(10 \text{ fm})^2} = 2.05 \text{ MeV}$$

$$(c) \Delta E_{21} = 3E_1 \quad (\text{See Problem 6-16})$$

$$\text{For the electron: } \Delta E_{21} = 3E_1 = 1.13 \times 10^4 \text{ MeV}$$

$$\text{For the proton: } \Delta E_{21} = 3E_1 = 6.15 \text{ MeV}$$

6-24. Refer to MORE section "Graphical Solution of the Finite Square Well". If there are only two allowed energies within the well, the highest energy $E_2 = V_0$, the

depth of the well. From Figure 6-14, $ka = \pi/2$, i.e., $ka = \frac{\sqrt{2mE_2}}{\hbar} \times a = \pi/2$

where $a = 1/2(1.0 \text{ fm}) = 0.5 \text{ fm}$ and $m = 939.6 \text{ MeV}/c^2$ for the neutron.

Substituting above, squaring, and re-arranging, we have:

$$E_2 = V_0 = \left(\frac{\pi}{2}\right)^2 \frac{\hbar^2}{2(939.6 \text{ MeV}/c^2)(0.5 \text{ fm})^2}$$

$$V_0 = \frac{(\pi)^2 (\hbar c)^2}{8(939.6 \text{ MeV})(0.5 \text{ fm} \times 10^{-6} \text{ nm}/\text{fm})^2} = \frac{(\pi)^2 (197.3 \text{ eV} \cdot \text{nm})^2}{8(939.6 \times 10^6 \text{ eV})(0.5 \times 10^{-6} \text{ nm})^2}$$

$$V_0 = 2.04 \times 10^8 \text{ eV} = 204 \text{ MeV}$$

$$6-28. \quad \langle p_x \rangle = \int_{-\infty}^{+\infty} \psi_3^*(x) \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \psi_3(x) dx \quad (\text{Equation 6-48})$$

$$= \int_0^L \sqrt{\frac{2}{L}} \sin \frac{3\pi x}{L} \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \sqrt{\frac{2}{L}} \sin \frac{3\pi x}{L} dx$$

$$= \frac{2}{L} \frac{\hbar}{i} \int_0^L \left(\sin \frac{3\pi x}{L} \right) \left(\cos \frac{3\pi x}{L} \right) \left(\frac{3\pi}{L} \right) dx$$

Let

$$\frac{3\pi x}{L} = y \quad \text{Then } x = 0 \rightarrow y = 0, \quad x = L \rightarrow y = 3\pi, \text{ and } \frac{3\pi}{L} dx = dy \rightarrow dx = \frac{L}{3\pi} dy$$

Substituting above gives:

$$\begin{aligned}\langle p_x \rangle &= \frac{2}{L} \frac{\hbar}{i} \frac{L}{3\pi} \int_0^{3\pi} \sin y \cos y dy \times \left(\frac{3\pi}{L} \right) \\ &= \frac{2}{L} \frac{\hbar}{i} \int_0^{3\pi} \sin y \cos y dy \\ &= \frac{2}{L} \frac{\hbar}{i} \left(\frac{\sin^2 y}{2} \right) \Big|_0^{3\pi} = \frac{2}{L} \frac{\hbar}{i} (0 - 0) = 0\end{aligned}$$

Reconciliation: p_x is a vector pointing half the time in the $+x$ direction, half in the $-x$ direction. E_k is a scalar proportional to v^2 , hence always positive.

6-29. For $n = 3$, $\psi_3 = (2/L)^{1/2} \sin(3\pi x/L)$

(a) $\langle x \rangle = \int_0^L x (2/L) \sin^2(3\pi x/L) dx$

Substituting $u = 3\pi x/L$, then $x = Lu/3\pi$ and $dx = (L/3\pi) du$. The limits

become:

$$x = 0 \rightarrow u = 0 \text{ and } x = L \rightarrow u = 3\pi$$

$$\begin{aligned}\langle x \rangle &= (2/L)(L/3\pi)(1/3\pi) \int_0^{3\pi} u \sin^2 u du \\ &= (2/L)(L/3\pi)^2 \left[\frac{u^2}{4} - \frac{u \sin 2u}{4} - \frac{\cos 2u}{8} \right]_0^{3\pi} \\ &= (2/L)(1/3\pi)^2 (3\pi)^2 / 4 = L/2\end{aligned}$$

(b) $\langle x^2 \rangle = \int_0^L x^2 (2/L) \sin^2(3\pi x/L) dx$

Changing the variable exactly as in (a) and noting that:

$$\int_0^{3\pi} u^2 \sin^2 u du = \left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8} \right) \sin 2u - \frac{u \cos 2u}{4} \right]_0^{3\pi}$$

$$\text{We obtain } \langle x^2 \rangle = \left(\frac{1}{3} - \frac{1}{8} \pi^2 \right) L^2 = 0.320L^2$$

$$6-31. \quad -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \quad (\text{Equation 6-18})$$

$$\frac{1}{2m} \left(\frac{\hbar}{i} \frac{d}{dx} \right) \left(\frac{\hbar}{i} \frac{d}{dx} \right) \psi(x) = [E - V(x)]\psi(x)$$

$$\frac{1}{2m} p_{op} p_{op} \psi = [E - V(x)]\psi$$

Multiplying by ψ^* and integrating over the range of x ,

$$\int_{-\infty}^{+\infty} \psi^* \frac{p_{op}^2}{2m} \psi dx = \int_{-\infty}^{+\infty} \psi^* [E - V(x)]\psi dx$$

$$\left\langle \frac{p^2}{2m} \right\rangle = \langle [E - V(x)] \rangle \quad \text{or} \quad \langle p^2 \rangle = \langle 2m [E - V(x)] \rangle$$

For the infinite square well $V(x) = 0$ wherever $\psi(x)$ does not vanish and vice versa.

$$\text{Thus, } \langle V(x) \rangle = 0 \text{ and } \langle p^2 \rangle = \langle 2mE \rangle = \left\langle 2m \frac{n^2 \pi^2 \hbar^2}{2mL^2} \right\rangle = \frac{\pi^2 \hbar^2}{L^2} \text{ for } n = 1$$

$$6-32. \quad \langle x^2 \rangle = \frac{L^2}{3} - \frac{L^2}{2\pi^2} \quad (\text{See Problem 6-29.}) \quad \text{And } \langle x \rangle = \frac{L}{2}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \left[\frac{L^2}{3} - \frac{L^2}{2\pi^2} - \frac{L^2}{4} \right]^{1/2} = L \left[\frac{1}{12} - \frac{1}{2\pi^2} \right]^{1/2} = 0.181L$$

$$\langle p_x^2 \rangle = \frac{\pi^2 \hbar^2}{L^2} \quad \text{and} \quad \langle p \rangle = 0 \quad (\text{See Problem 6-31})$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \left[\frac{\pi^2 \hbar^2}{L^2} - 0 \right]^{1/2} = \frac{\pi \hbar}{L}. \quad \text{And } \sigma_x \sigma_p = (0.181L)(\pi \hbar / L) = 0.568\hbar$$

$$6-33. \quad \psi_0(x) = A_0 e^{-m\omega x^2 / 2\hbar} \quad \text{where } A_0 = (m\omega / \hbar \pi)^{1/4}$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} A_0^2 x e^{-m\omega x^2 / \hbar} dx \quad \text{Letting } u^2 = m\omega x^2 / \hbar \text{ and } x = (\hbar / m\omega)^{1/2} u$$

$$2u du = (m\omega / \hbar)(2x dx). \quad \text{And thus, } (m\omega / \hbar)^{-1} u du = x dx; \text{ limits are unchanged.}$$

$$\langle x \rangle = A_0^2 (\hbar / m\omega) \int_{-\infty}^{+\infty} u e^{-u^2} du = 0 \quad (\text{Note that the symmetry of } V(x) \text{ would also tell}$$

us that

$$\langle x \rangle = 0.)$$

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{+\infty} A_0^2 x^2 e^{-m\omega x^2 / \hbar} dx \\ &= A_0^2 (\hbar / m\omega)^{3/2} \int_{-\infty}^{+\infty} u^2 e^{-u^2} du = 2A_0^2 (\hbar / m\omega)^{3/2} \int_{-\infty}^{+\infty} u^2 e^{-u^2} du \\ &= 2A_0^2 (\hbar / m\omega)^{3/2} \sqrt{\pi} / 4 = (m\omega / \hbar\pi)^{1/2} (\hbar / m\omega)^{3/2} \sqrt{\pi} / 2 = \hbar / (2m\omega) \end{aligned}$$

6-34. $\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = (n + 1/2)\hbar\omega$. For the ground state ($n = 0$),

$$\langle x^2 \rangle = \frac{2}{m\omega^2} (\hbar\omega / 2 - p^2 / 2m) \quad \text{and} \quad \langle x^2 \rangle = \left\langle \frac{\hbar}{m\omega} - \frac{p^2}{m^2\omega^2} \right\rangle = \hbar / 2m\omega \quad (\text{See}$$

Problem 6-33)

$$\frac{\hbar}{m\omega} \left\langle 1 - \frac{p^2}{m\hbar\omega} \right\rangle = \frac{\hbar}{2m\omega} \quad \text{or} \quad \left\langle 1 - \frac{p^2}{m\hbar\omega} \right\rangle = \frac{1}{2} \rightarrow \langle p^2 \rangle = \frac{1}{2} m\hbar\omega$$

6-35. (a) $\Psi_0(x, t) = (m\omega / \hbar\pi)^{1/4} e^{-m\omega x^2 / 2\hbar} e^{-i\omega t / 2}$

(b) $p_{xop} = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad \langle p^2 \rangle = \int_{-\infty}^{+\infty} \Psi_0^*(x, t) \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \Psi_0(x, t) dx$

$$\frac{\partial \Psi_0}{\partial x} = A_0 (m\omega x / \hbar\pi)^{1/4} e^{-m\omega x^2 / 2\hbar} e^{-i\omega t / 2}$$

$$\frac{\partial^2 \Psi_0}{\partial x^2} = A_0 [(-m\omega x / \hbar)(-m\omega x / \hbar) - m\omega / \hbar] e^{-m\omega x^2 / 2\hbar} e^{-i\omega t / 2}$$

$$\langle p^2 \rangle = -\hbar^2 A_0^2 (m\omega / \hbar) \int_{-\infty}^{+\infty} (m\omega x^2 / \hbar - 1) e^{-m\omega x^2 / \hbar} dx$$

$$= -\hbar^2 A_0^2 (m\omega / \hbar) \left[\int_{-\infty}^{+\infty} (m\omega x^2 / \hbar) e^{-m\omega x^2 / \hbar} dx - \int_{-\infty}^{+\infty} e^{-m\omega x^2 / \hbar} dx \right]$$

Letting $u = (m\omega x / \hbar)^{1/2} x$, then

$$\begin{aligned}
 \langle p^2 \rangle &= -\hbar^2 A_0^2 (m\omega / \hbar) (m\omega \hbar)^{-1/2} \left[\int_{-\infty}^{+\infty} u^2 e^{-u^2} du - \int_{-\infty}^{+\infty} e^{-u^2} du \right] \\
 &= -\hbar^2 A_0^2 (m\omega / \hbar)^{1/2} 2 \left[\int_0^{\infty} u^2 e^{-u^2} du - \int_0^{\infty} e^{-u^2} du \right] \\
 &= -\hbar^2 (m\omega / \hbar \pi)^{1/2} (m\omega / \hbar)^{1/2} 2 \left(\frac{\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} \right) \\
 &= \hbar^2 (m\omega / \hbar) (1/2) = m\hbar\omega / 2
 \end{aligned}$$

6-36. $\psi_0(x) = C_0 e^{-m\omega x^2 / 2\hbar}$ (Equation 6-58)

$$\begin{aligned}
 \text{(a)} \quad \int_{-\infty}^{+\infty} |\psi_0(x)|^2 dx &= 1 = \int_{-\infty}^{+\infty} |C_0|^2 e^{-m\omega x^2 / \hbar} dx \\
 &= |C_0|^2 \times 2I_0 = |C_0|^2 \times 2 \times \frac{1}{2} \sqrt{\frac{\pi}{\lambda}} \quad \text{with } \lambda = m\omega / \hbar \\
 &= |C_0|^2 \sqrt{\frac{\pi \hbar}{m\omega}}
 \end{aligned}$$

$$C_0 = \left(\frac{m\omega}{\pi \hbar} \right)^{1/4}$$

$$\begin{aligned}
 \text{(b)} \quad \langle x^2 \rangle &= \int_{-\infty}^{+\infty} x^2 |\psi_0|^2 dx = \int_{-\infty}^{+\infty} x^2 \sqrt{\frac{m\omega}{\pi \hbar}} e^{-m\omega x^2 / \hbar} dx \\
 &= \sqrt{\frac{m\omega}{\pi \hbar}} \times 2I_2 = \sqrt{\frac{m\omega}{\pi \hbar}} \times 2 \times \frac{1}{4} \sqrt{\frac{\pi}{\lambda^3}} \quad \text{with } \lambda = m\omega / \hbar \\
 &= \frac{1}{2} \sqrt{\frac{m\omega}{\pi \hbar}} \sqrt{\frac{\pi \hbar^3}{m^3 \omega^3}} = \frac{1}{2} \frac{\hbar}{m\omega}
 \end{aligned}$$

$$\text{(c)} \quad \langle V(x) \rangle = \left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{1}{2} m\omega \times \frac{1}{2} \frac{\hbar}{m\omega} = \frac{1}{4} \hbar\omega$$

6-37. $\psi_1(x) = C_1 x e^{-m\omega x^2/2\hbar}$ (Equation 6-58)

(a) $\int_{-\infty}^{+\infty} |\psi_1(x)|^2 dx = 1 = \int_{-\infty}^{+\infty} |C_1|^2 x^2 e^{-m\omega x^2/\hbar} dx = |C_1|^2 \times 2I_2$

$$= |C_1|^2 \times 2 \times \frac{1}{4} \sqrt{\frac{\pi}{\lambda^3}} \quad \text{with } \lambda = m\omega/\hbar$$

$$= |C_1|^2 \times \frac{1}{2} \sqrt{\frac{\pi \hbar^3}{m^3 \omega^3}}$$

$$C_1 = \left(\frac{4m^3 \omega^3}{\pi \hbar^3} \right)^{1/4}$$

(b) $\langle x \rangle = \int_{-\infty}^{+\infty} x |\psi_1|^2 dx = \int_{-\infty}^{+\infty} x^3 \left(\frac{4m^3 \omega^3}{\pi \hbar^3} \right)^{1/2} e^{-m\omega x^2/\hbar} dx = 0$

(c) $\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_1|^2 dx = \int_{-\infty}^{+\infty} x^2 \left(\frac{4m^3 \omega^3}{\pi \hbar^3} \right)^{1/2} e^{-m\omega x^2/\hbar} x^2 dx$

$$= \left(\frac{4m^3 \omega^3}{\pi \hbar^3} \right)^{1/2} \times 2I_4 = \left(\frac{4m^3 \omega^3}{\pi \hbar^3} \right)^{1/2} \times 2 \times \frac{3}{8} \sqrt{\frac{\pi}{\lambda^5}} \quad \text{where } \lambda = m\omega/\hbar$$

$$= \frac{3}{2} \sqrt{\frac{m^3 \omega^3}{\pi \hbar^3}} \sqrt{\frac{\pi \hbar^5}{m^5 \omega^5}} = \frac{3}{2} \frac{\hbar}{m\omega}$$

(d) $\langle V(x) \rangle = \left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{1}{2} m\omega^2 \times \frac{3}{2} \frac{\hbar}{m\omega} = \frac{3}{4} \hbar\omega$

6-42. $\psi_0(x) = A_0 e^{-\omega x^2/2\hbar}$ $\psi_1(x) = A_1 \sqrt{\frac{m\omega}{\hbar}} x e^{-m\omega x^2/2\hbar}$

From Equation 6-58.

Note that ψ_0 is an even function of x and ψ_1 is an odd function of x .

It follows that $\int_{-\infty}^{+\infty} \psi_0 \psi_1 dx = 0$

6-51. (a) The probability density for the ground state is

$$P(x) = \psi^2(x) = (2/L) \sin^2 \pi x/L.$$

The probability of finding the particle in the range $0 < x < L/2$ is:

$$P = \int_0^{L/2} P(x) dx = \frac{2}{L} \frac{L}{\pi} \int_0^{\pi/2} \sin^2 u du = \frac{2}{\pi} \left(\frac{\pi}{4} - 0 \right) = \frac{1}{2} \quad \text{where } u = \pi x / L$$

$$(b) \quad P = \int_0^{L/3} P(x) dx = \frac{2}{L} \frac{L}{\pi} \int_0^{\pi/3} \sin^2 u du = \frac{2}{\pi} \left(\frac{\pi}{6} - \frac{\sin 2\pi/3}{4} \right) = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.195$$

(Note: 1/3 is the classical result.)

$$(c) \quad P = \int_0^{3L/4} P(x) dx = \frac{2}{L} \frac{L}{\pi} \int_0^{3\pi/4} \sin^2 u du = \frac{2}{\pi} \left(\frac{3\pi}{8} - \frac{\sin 3\pi/2}{4} \right) = \frac{3}{4} + \frac{1}{2\pi} = 0.909$$

(Note: 3/4 is the classical result.)