PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #5 SOLUTIONS

(1) You know that at most one fermion may occupy any given single-particle state. A *parafermion* is a particle for which the maximum occupancy of any given single-particle state is k, where k is an integer greater than zero. (For $k = 1$, parafermions are regular everyday fermions; for $k = \infty$, parafermions are regular everyday bosons.) Consider a system with one single-particle level whose energy is ε, *i.e.* the Hamiltonian is simply $\mathcal{H} = \varepsilon\, n$, where n is the particle number.

(a) Compute the partition function $\Xi(\mu, T)$ in the grand canonical ensemble for parafermions.

(b) Compute the occupation function $n(\mu, T)$. What is n when $\mu = -\infty$? When $\mu = \varepsilon$? When $\mu = +\infty$? Does this make sense? Show that $n(\mu, T)$ reduces to the Fermi and Bose distributions in the appropriate limits.

(c) Sketch $n(\mu, T)$ as a function of μ for both $T = 0$ and $T > 0$.

(d) Can a gas of ideal parafermions condense in the sense of Bose condensation?

Solution : The general expression for Ξ is

$$\Xi = \prod_\alpha \sum_{n_\alpha} \left(z\, e^{-\beta\varepsilon_\alpha}\right)^{n_\alpha} \ .$$

Now the sum on n runs from 0 to k, and

$$\sum_{n=0}^{k} x^n = \frac{1 - x^{k+1}}{1 - x} \ .$$

(a) Thus,

$$\Xi = \frac{1 - e^{(k+1)\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)}} \ .$$

(b) We then have

$$\begin{aligned} n &= -\frac{\partial\Omega}{\partial\mu} = \frac{1}{\beta}\frac{\partial \ln \Xi}{\partial\mu} \\ &= \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} - \frac{k+1}{e^{(k+1)\beta(\varepsilon-\mu)} - 1} \end{aligned}$$

(c) A plot of $n(\varepsilon, T, \mu)$ for $k = 3$ is shown in fig. 1. Qualitatively the shape is that of the Fermi function $f(\varepsilon - \mu)$. At $T = 0$, the occupation function is $n(\varepsilon, T = 0, \mu) = k\,\Theta(\mu - \varepsilon)$. This step function smooths out for T finite.

(d) For each $k < \infty$, the occupation number $n(z, T)$ is a finite order polynomial in z, and hence an analytic function of z. Therefore, there is no possibility for Bose condensation except for $k = \infty$.

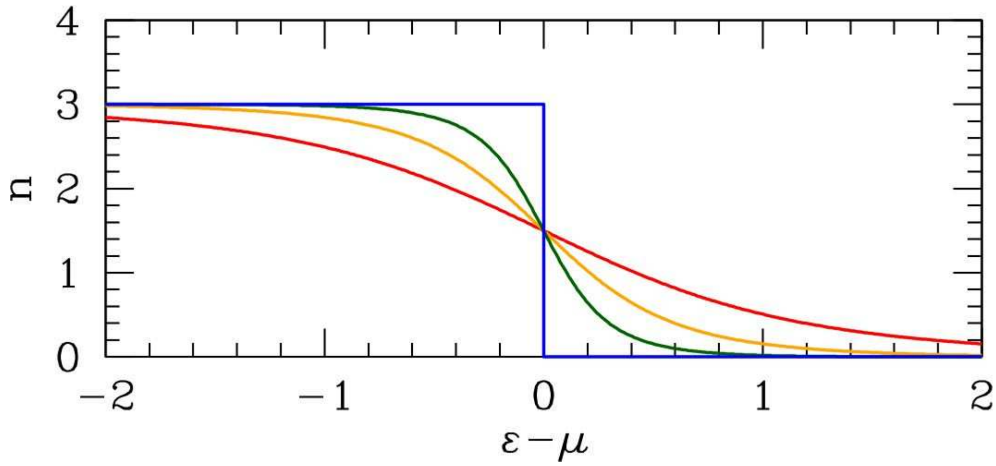


Figure 1: (3)(c) $k = 3$ parafermion occupation number *versus* $\varepsilon - \mu$ for $k_B T = 0$, $k_B T = 0.25$, $k_B T = 0.5$, and $k_B T = 1$.

(2) Consider a system of N spin-$\frac{1}{2}$ particles occupying a volume V at temperature T. Opposite spin fermions may bind in a singlet state to form a boson:

$$\mathrm{f}\!\uparrow \; + \; \mathrm{f}\!\downarrow \quad \rightleftharpoons \quad \mathrm{b}$$

with a binding energy $-\Delta < 0$. Assume that all the particles are nonrelativistic; the fermion mass is m and the boson mass is $2m$. Assume further that spin-flip processes exist, so that the $\uparrow$ and $\downarrow$ fermion species have identical chemical potential μ_{f}.

(a) What is the equilibrium value of the boson chemical potential, μ_{b}? *Hint* : the answer is $\mu_{\mathrm{b}} = 2\mu_{\mathrm{f}}$.

(b) Let the total mass density be ρ. Derive the equation of state $\rho = \rho(\mu_{\mathrm{f}}, T)$, assuming the bosons have not condensed. You may wish to abbreviate

$$\zeta_p(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^p} \; .$$

(c) At what value of μ_{f} do the bosons condense?

(d) Derive an equation for the Bose condensation temperature T_{c}. Solve for T_{c} in the limits $\varepsilon_0 \ll \Delta$ and $\varepsilon_0 \gg \Delta$, respectively, where

$$\varepsilon_0 \equiv \frac{\pi\hbar^2}{m} \left(\frac{\rho/2m}{\zeta\big(\frac{3}{2}\big)} \right)^{2/3} \; .$$

(e) What is the equation for the condensate fraction $\rho_0(T,\rho)/\rho$ when $T < T_{\mathrm{c}}$?

Solution :

(a) The chemical potential is the Gibbs free energy per particle. If the fermion and boson species are to coexist at the same T and p, the reaction $\mathrm{f}\!\uparrow + \mathrm{f}\!\downarrow \longrightarrow \mathrm{b}$ must result in $\Delta G = \mu_\mathrm{b} - 2\mu_\mathrm{f} = 0$.

(b) For $T > T_\mathrm{c}$,

$$\rho = -2m\,\lambda_T^{-3}\,\zeta_{3/2}\big(-\,e^{\mu_\mathrm{f}/k_\mathrm{B}T}\big) + 2\sqrt{8}\,m\,\lambda_T^{-3}\,\zeta_{3/2}\big(e^{(2\mu_\mathrm{f}+\Delta)/k_\mathrm{B}T}\big) \ ,$$

where $\lambda_T = \sqrt{2\pi\hbar^2/mk_\mathrm{B}T}$ is the thermal wavelength for particles of mass m. This formula accounts for both fermion spin polarizations, each with number density $n_{\mathrm{f}\uparrow} = n_{\mathrm{f}\downarrow} = -\lambda_T^{-3}\,\zeta_{3/2}(-z_\mathrm{f})$ and the bosons with number density $\sqrt{8}\,\lambda_T^{-3}\,\zeta_{3/2}(z_\mathrm{b}\,e^{\beta\Delta})$, with $z_\mathrm{b} = z_\mathrm{f}^2$ due to chemical equilibrium among the species. The factor of $2^{3/2} = \sqrt{8}$ arises from the fact that the boson mass is $2m$, hence the boson thermal wavelength is $\lambda_T/\sqrt{2}$.

(c) The bosons condense when $\mu_\mathrm{b} = -\Delta$, the minimum single particle energy. This means $\mu_\mathrm{f} = -\frac{1}{2}\Delta$. The equation of state for $T < T_\mathrm{c}$ is then

$$\rho = -2m\,\lambda_T^{-3}\,\zeta_{3/2}\big(-\,e^{-\Delta/2k_\mathrm{B}T}\big) + 4\sqrt{2}\,\zeta\big(\tfrac{3}{2}\big)\,m\,\lambda_T^{-3} + \rho_0 \ ,$$

where ρ_0 is the condensate mass density.

(d) At $T = T_\mathrm{c}$ we have $\rho_0 = 0$, hence

$$\frac{\rho}{2m}\left(\frac{2\pi\hbar^2}{mk_\mathrm{B}T_\mathrm{c}}\right)^{3/2} = \sqrt{8}\,\zeta\big(\tfrac{3}{2}\big) - \zeta_{3/2}\big(-\,e^{-\Delta/2k_\mathrm{B}T_\mathrm{c}}\big) \ ,$$

which is a transcendental equation. Om. In the limit where Δ is very large, we have

$$T_\mathrm{c}(\Delta \gg \varepsilon_0) = \frac{\pi\hbar^2}{mk_\mathrm{B}}\left(\frac{\rho/2m}{\zeta\big(\frac{3}{2}\big)}\right)^{2/3} = \frac{\varepsilon_0}{k_\mathrm{B}} \ .$$

In the opposite limit, we have $\Delta \to 0^+$ and $-\zeta_{3/2}(-1) = \eta(3/2)$, where $\eta(s)$ is the Dirichlet η-function,

$$\eta(s) = \sum_{j=1}^{\infty}(-1)^{j-1}\,j^{-s} = \big(1 - 2^{1-s}\big)\,\zeta(s) \ .$$

Then

$$T_\mathrm{c}(\Delta \ll \varepsilon_0) = \frac{2\varepsilon_0/k_\mathrm{B}}{\big(1 + \frac{3}{2}\sqrt{2}\big)^{2/3}} \ .$$

(e) The condensate fraction is

$$\nu = \frac{\rho_0}{\rho} = 1 - \left(\frac{T}{T_\mathrm{c}}\right)^{3/2} \cdot \ \frac{\sqrt{8}\,\zeta\big(\frac{3}{2}\big) - \zeta_{3/2}\big(-\,e^{-\Delta/2k_\mathrm{B}T}\big)}{\sqrt{8}\,\zeta\big(\frac{3}{2}\big) - \zeta_{3/2}\big(-\,e^{-\Delta/2k_\mathrm{B}T_\mathrm{c}}\big)} \ .$$

Note that as $\Delta \to -\infty$ we have $-\zeta_{3/2}\big(-\,e^{-\Delta/2k_\mathrm{B}T}\big) \to 0$ and the condensate fraction approaches the free boson result, $\nu = 1 - (T/T_\mathrm{c})^{3/2}$. In this limit there are no fermions present.

(3) A three-dimensional system of spin-0 bosonic particles obeys the dispersion relation

$$\varepsilon(\boldsymbol{k}) = \Delta + \frac{\hbar^2 \boldsymbol{k}^2}{2m} \ .$$

The quantity Δ is the formation energy and m the mass of each particle. These particles are not conserved – they may be created and destroyed at the boundaries of their environment. (A possible example: vacancies in a crystalline lattice.) The Hamiltonian for these particles is

$$\mathcal{H} = \sum_{\boldsymbol{k}} \varepsilon(\boldsymbol{k})\, \hat{n}_{\boldsymbol{k}} + \frac{U}{2V}\, \hat{N}^2 \ ,$$

where $\hat{n}_{\boldsymbol{k}}$ is the number operator for particles with wavevector $\boldsymbol{k}$, $\hat{N} = \sum_{\boldsymbol{k}} \hat{n}_{\boldsymbol{k}}$ is the total number of particles, V is the volume of the system, and U is an interaction potential.

(a) Treat the interaction term within mean field theory. That is, define $\hat{N} = \langle \hat{N} \rangle + \delta \hat{N}$, where $\langle \hat{N} \rangle$ is the thermodynamic average of $\hat{N}$, and derive the mean field self-consistency equation for the number density $\rho = \langle \hat{N} \rangle / V$ by neglecting terms quadratic in the fluctuations $\delta \hat{N}$. Show that the mean field Hamiltonian is

$$\mathcal{H}_{\text{MF}} = -\tfrac{1}{2} V U \rho^2 + \sum_{\boldsymbol{k}} \Big[\varepsilon(\boldsymbol{k}) + U\rho \Big]\, \hat{n}_{\boldsymbol{k}} \ ,$$

(b) Derive the criterion for Bose condensation. Show that this requires $\Delta < 0$. Find an equation relating T_{c}, U, and Δ.

Solution :

(a) We write

$$\begin{aligned} \hat{N}^2 &= \big(\langle \hat{N} \rangle + \delta \hat{N} \big)^2 \\ &= \langle \hat{N} \rangle^2 + 2 \langle \hat{N} \rangle\, \delta \hat{N} + (\delta \hat{N})^2 \\ &= -\langle \hat{N} \rangle^2 + 2 \langle \hat{N} \rangle\, \hat{N} + (\delta \hat{N})^2 \ . \end{aligned}$$

We drop the last term, $(\delta \hat{N})^2$, because it is quadratic in the fluctuations. This is the mean field assumption. The Hamiltonian now becomes

$$\mathcal{H}_{\text{MF}} = -\tfrac{1}{2} V U \rho^2 + \sum_{\boldsymbol{k}} \Big[\varepsilon(\boldsymbol{k}) + U\rho \Big]\, \hat{n}_{\boldsymbol{k}} \ ,$$

where $\rho = \langle \hat{N} \rangle / V$ is the number density. This, the dispersion is effectively changed, to

$$\tilde{\varepsilon}(\boldsymbol{k}) = \frac{\hbar^2 \boldsymbol{k}^2}{2m} + \Delta + U\rho \ .$$

The average number of particles in state $\big|\, \boldsymbol{k} \,\big\rangle$ is given by the Bose function,

$$\langle \hat{n}_{\boldsymbol{k}} \rangle = \frac{1}{\exp\big[\tilde{\varepsilon}(\boldsymbol{k}) / k_{\text{B}} T \big] - 1} \ .$$

Summing over all $\boldsymbol{k}$ states, and using

$$\frac{1}{V}\sum_{\boldsymbol{k}} \longrightarrow \int\!\frac{d^3\!k}{(2\pi)^3} \ ,$$

we obtain

$$\begin{aligned}
\rho &= \frac{1}{V}\sum_{\boldsymbol{k}} \langle \hat{n}_{\boldsymbol{k}} \rangle \\
&= \rho_0 + \int\!\frac{d^3\!k}{(2\pi)^3}\, \frac{1}{e^{\hbar^2 k^2/2mk_{\rm B}T}\, e^{(\Delta+U\rho)/k_{\rm B}T} - 1} \\
&= \rho_0 + \int\limits_0^\infty\! d\varepsilon\, \frac{g(\varepsilon)}{e^{(\varepsilon+\Delta+U\rho)/k_{\rm B}T} - 1}
\end{aligned}$$

where $\rho_0 = \langle \hat{n}_{\boldsymbol{k}=0}\rangle/V$ is the number density of the $\boldsymbol{k}=0$ state alone, *i.e.* the condensate density. When there is no condensate, $\rho_0 = 0$. The above equation is the mean field equation. It is equivalent to demanding $\partial F/\partial\rho = 0$, *i.e.* to extremizing the free energy with respect to the mean field parameter ρ. Though it is not a required part of the solution, we have here written this relation in terms of the density of states $g(\varepsilon)$, defined according to

$$g(\varepsilon) \equiv \int\!\frac{d^3\!k}{(2\pi)^3}\, \delta\bigg(\varepsilon - \frac{\hbar^2\boldsymbol{k}^2}{2m}\bigg) = \frac{m^{3/2}}{\sqrt{2}\,\pi^2\hbar^3}\,\sqrt{\varepsilon} \ .$$

(b) Bose condensation requires

$$\Delta + U\rho = 0 \ ,$$

which clearly requires $\Delta < 0$. Writing $\Delta = -|\Delta|$, we have, just at $T = T_{\rm c}$,

$$\rho(T_{\rm c}) = \frac{|\Delta|}{U} = \int\!\frac{d^3\!k}{(2\pi)^3}\, \frac{1}{e^{\hbar^2 k^2/2mk_{\rm B}T_{\rm c}} - 1} \ ,$$

since $\rho_0(T_{\rm c}) = 0$. This relation determines $T_{\rm c}$. Explicitly, we have

$$\begin{aligned}
\frac{|\Delta|}{U} &= \int\limits_0^\infty\! d\varepsilon\, g(\varepsilon) \sum_{j=1}^\infty e^{-j\varepsilon/k_{\rm B}T_{\rm c}} \\
&= \zeta\big(\tfrac{3}{2}\big)\bigg(\frac{mk_{\rm B}T_c}{2\pi\hbar^2}\bigg)^{3/2} \ ,
\end{aligned}$$

where $\zeta(\ell) = \sum_{n=1}^\infty n^{-\ell}$ is the Riemann zeta function. Thus,

$$T_{\rm c} = \frac{2\pi\hbar^2}{mk_{\rm B}} \left(\frac{|\Delta|}{\zeta\big(\frac{3}{2}\big)\,U}\right)^{2/3} \ .$$

(4) The $n^{\rm th}$ moment of the normalized Gaussian distribution $P(x) = (2\pi)^{-1/2} \exp\big(-\frac{1}{2}x^2\big)$ is defined by

$$\langle x^n \rangle = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \! dx \; x^n \; \exp\big(-\tfrac{1}{2}x^2\big)$$

Clearly $\langle x^n \rangle = 0$ if n is a nonnegative odd integer. Next consider the *generating function*

$$Z(j) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \! dx \; \exp\big(-\tfrac{1}{2}x^2\big) \exp(jx) = \exp\big(\tfrac{1}{2}j^2\big) \; .$$

(a) Show that

$$\langle x^n \rangle = \frac{d^n Z}{dj^n}\bigg|_{j=0}$$

and provide an explicit result for $\langle x^{2k} \rangle$ where $k \in \mathbb{N}$.

(b) Now consider the following integral:

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \! dx \; \exp\Big(-\tfrac{1}{2}x^2 - \tfrac{1}{4!}\lambda\, x^4\Big) \; .$$

This has no analytic solution but we may express the result as a power series in the parameter λ by Taylor expanding $\exp\big(-\frac{\lambda}{4!}\, x^4\big)$ and then using the result of part (a) for the moments $\langle x^{4k} \rangle$. Find the coefficients in the perturbation expansion,

$$F(\lambda) = \sum_{k=0}^{\infty} C_k \, \lambda^k \; .$$

(c) Define the *remainder after* N *terms* as

$$R_N(\lambda) = F(\lambda) - \sum_{k=0}^{N} C_k \, \lambda^k \; .$$

Compute $R_N(\lambda)$ by evaluating numerically the integral for $F(\lambda)$ (using `Mathematica` or some other numerical package) and subtracting the finite sum. Then define the ratio $S_N(\lambda) = R_N(\lambda)/F(\lambda)$, which is the relative error from the N term approximation and plot the absolute relative error $\big|S_N(\lambda)\big|$ *versus* N for several values of λ. (I suggest you plot the error on a log scale.) What do you find?? Try a few values of λ including $\lambda = 0.01$, $\lambda = 0.05$, $\lambda = 0.2$, $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$.

Solution :

(a) Clearly

$$\frac{d^n}{dj^n}\bigg|_{j=0} e^{jx} = x^n \; ,$$

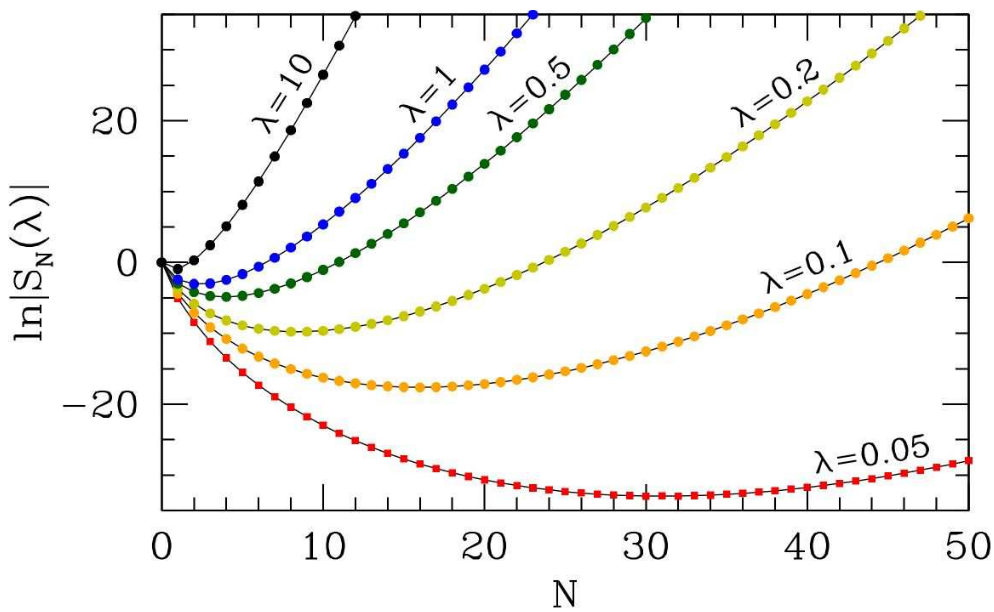


Figure 2: (5)(c) Relative error *versus* number of terms kept for the asymptotic series for $F(\lambda)$. Note that the optimal number of terms to sum is $N^*(\lambda) \approx \frac{3}{2\lambda}$.

so $\langle\, x^n \,\rangle = \big(d^n Z/dj^n\big)_{j=0}$. With $Z(j) = \exp\big(\frac{1}{2}j^2\big)$, only the $k^{\rm th}$ order term in j^2 in the Taylor series for $Z(j)$ contributes, and we obtain

$$\langle\, x^{2k} \,\rangle = \frac{d^{2k}}{dj^{2k}} \left(\frac{j^{2k}}{2^k\, k!} \right) = \frac{(2k)!}{2^k\, k!} \ .$$

(b) We have

$$F(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!} \right)^{\! n} \langle\, x^{4n} \,\rangle = \sum_{n=0}^{\infty} \frac{(4n)!}{4^n\,(4!)^n\, n!\,(2n)!}\,(-\lambda)^n \ .$$

This series is *asymptotic*. It has the properties

$$\lim_{\lambda\to 0} \frac{R_N(\lambda)}{\lambda^N} = 0 \quad (\text{fixed } N) \qquad , \qquad \lim_{N\to\infty} \frac{R_N(\lambda)}{\lambda^N} = \infty \quad (\text{fixed } \lambda) \ ,$$

where $R_N(\lambda)$ is the remainder after N terms, defined in part (c). The radius of convergence is zero. To see this, note that if we reverse the sign of λ, then the integrand of $F(\lambda)$ diverges badly as $x \to \pm\infty$. So $F(\lambda)$ is infinite for $\lambda < 0$, which means that there is no disk of any finite radius of convergence which encloses the point $\lambda = 0$. Note that by Stirling's rule,

$$C_n \equiv \frac{(4n)!}{4^n\,(4!)^n\, n!\,(2n)!} \sim n^n \cdot \big(\tfrac{2}{3}\big)^n\, e^{-n} \cdot (\pi n)^{-1/2} \ ,$$

and we conclude that the magnitude of the summand reaches a minimum value when $n = n^*(\lambda)$, with

$$n^*(\lambda) \approx \frac{3}{2\lambda}$$

for small values of λ. For large n, the coefficient C_n grows as $C_n \sim e^{n \ln n + \mathcal{O}(n)}$, which dominates the $(-\lambda)^n$ term, no matter how small λ is.

(c) Results are plotted in fig. 2.

It is worth pointing out that the series for $F(\lambda)$ and for $\ln F(\lambda)$ have diagrammatic interpretations. For a Gaussian integral, one has

$$\langle \, x^{2k} \, \rangle = \langle \, x^2 \, \rangle^k \cdot A_{2k}$$

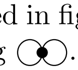
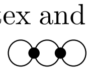
where A_{2k} is the *number of contractions*. For a proof, see §3.2.2 of the notes. For our integral, $\langle \, x^2 \, \rangle = 1$. The number of contractions A_{2k} is computed in the following way. For each of the $2k$ powers of x, we assign an index running from 1 to $2k$. The indices are *contracted*, *i.e.* paired, with each other. How many pairings are there? Suppose we start with any from among the $2k$ indices. Then there are $(2k-1)$ choices for its mate. We then choose another index arbitrarily. There are now $(2k-3)$ choices for *its* mate. Carrying this out to its completion, we find that the number of contractions is

$$A_{2k} = (2k-1)(2k-3) \cdots 3 \cdot 1 = \frac{(2k)!}{2^k \, k!} \ ,$$

exactly as we found in part (a). Now consider the integral $F(\lambda)$. If we expand the quartic term in a power series, then each power of λ brings an additional four powers of x. It is therefore convenient to represent each such quartet with the symbol $\times$. At order N of the series expansion, we have N $\times$'s and $4N$ indices to contract. Each full contraction of the indices may be represented as a labeled diagram, which is in general composed of several disjoint connected subdiagrams. Let us label these subdiagrams, which we will call clusters, by an index γ. Now suppose we have a diagram consisting of m_γ subdiagrams of type γ, for each γ. If the cluster γ contains n_γ vertices ($\times$), then we must have

$$N = \sum_\gamma m_\gamma \, n_\gamma \ .$$

How many ways are there of assigning the labels to such a diagram? One might think $(4!)^N \cdot N!$, since for each vertex $\times$ there are 4! permutations of its four labels, and there are $N!$ ways to permute all the vertices. However, this overcounts diagrams which are *invariant* under one or more of these permutations. We define the *symmetry factor* s_γ of the (unlabeled) cluster γ as the number of permutations of the indices of a corresponding labeled cluster which result in the same contraction. We can also permute among the m_γ identical disjoint clusters of type γ.

Examples of clusters and their corresponding symmetry factors are provided in fig. 3, for all diagrams with $n_\gamma \leq 3$. There is only one diagram with $n_\gamma = 1$, resembling . To obtain $s_\gamma = 8$, note that each of the circles can be separately rotated by an angle π about the long symmetry axis. In addition, the figure can undergo a planar rotation by π about an axis which runs through the sole vertex and is normal to the plane of the diagram. This results in $s_\gamma = 2 \cdot 2 \cdot 2 = 8$. For the cluster , there is one extra circle, so $s_\gamma = 2^4 = 16$. The third diagram in figure shows two vertices connected by four lines. Any of the 4! permutations

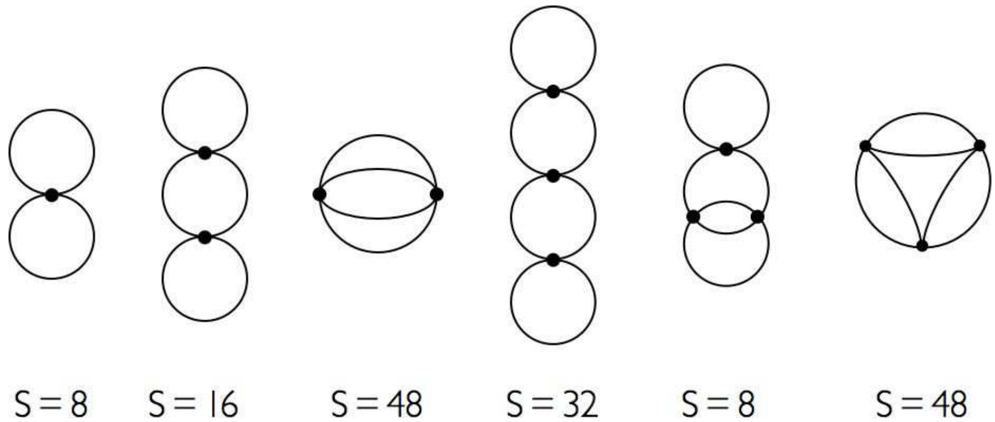


Figure 3: (5)(c) Cluster symmetry factors. A vertex is represented as a black dot (•) with four 'legs'.

of these lines results in the same diagram. In addition, we may reflect about the vertical symmetry axis, interchanging the vertices, to obtain another symmetry operation. Thus $s_\gamma = 2 \cdot 4! = 48$. One might ask why we don't also count the planar rotation by π as a symmetry operation. The answer is that it is equivalent to a combination of a reflection and a permutation, so it is not in fact a distinct symmetry operation. (If it were distinct, then s_γ would be 96.) Finally, consider the last diagram in the figure, which resembles a sausage with three links joined at the ends into a circle. If we keep the vertices fixed, there are 8 symmetry operations associated with the freedom to exchange the two lines associated with each of the three sausages. There are an additional 6 symmetry operations associated with permuting the three vertices, which can be classified as three in-plane rotations by 0, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, each of which can also be combined with a reflection about the y-axis (this is known as the group C_{3v}). Thus, $s_\gamma = 8 \cdot 6 = 48$.

Now let us compute an expression for $F(\gamma)$ in terms of the clusters. We sum over all possible numbers of clusters at each order:

$$\begin{aligned} F(\gamma) &= \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\{m_\gamma\}} \frac{(4!)^N N!}{\prod_\gamma s_\gamma^{m_\gamma} \, m_\gamma!} \left(-\frac{\lambda}{4!}\right)^N \delta_{N,\sum_\gamma m_\gamma n_\gamma} \\ &= \exp\left(\sum_\gamma \frac{(-\lambda)^{n_\gamma}}{s_\gamma}\right) . \end{aligned}$$

Thus,

$$\ln F(\gamma) = \sum_\gamma \frac{(-\lambda)^{n_\gamma}}{s_\gamma} ,$$

and *the logarithm of the sum over all diagrams is a sum over connected clusters.* It is instructive to work this out to order λ^2. We have, from the results of part (b),

$$F(\lambda) = 1 - \tfrac{1}{8}\,\lambda + \tfrac{35}{384}\,\lambda^2 + \mathcal{O}(\lambda^3) \quad \Longrightarrow \quad \ln F(\lambda) = -\tfrac{1}{8}\,\lambda + \tfrac{1}{12}\,\lambda^2 + \mathcal{O}(\lambda^3) .$$

Note that there is one diagram with $N = 1$ vertex, with symmetry factor $s = 8$. For $N = 2$ vertices, there are two diagrams, one with $s = 16$ and one with $s = 48$ (see fig. 3). Since $\frac{1}{16} + \frac{1}{48} = \frac{1}{12}$, the diagrammatic expansion is verified to order λ^2.

In quantum field theory (QFT), the vertices themselves carry space-time (or, more commonly, momentum-frequency) labels, and the contractions, *i.e.* the lines connecting legs of the vertices, are *propagators* $G(p_i^\mu - p_j^\mu)$, where p_i^μ is the 4-momentum associated with vertex i. We then must integrate over all the internal 4-momenta to obtain the numerical value for a given diagram. The diagrams, as you know, are associated with Feynman's approach to QFT and are known as Feynman diagrams. Our example here is equivalent to a $(0+0)$-dimensional field theory, *i.e.* zero space dimensions and zero time dimensions. There are then no internal 4-momenta to integrate over, and each propagator is simply a number rather than a function. The discussion above of symmetry factors s_γ carries over to the more general QFT case.

There is an important lesson to be learned here about the behavior of asymptotic series. As we have seen, if λ is sufficiently small, summing more and more terms in the perturbation series results in better and better results, until one reaches an optimal order when the error is minimized. Beyond this point, summing additional terms makes the result *worse*, and indeed the perturbation series diverges badly as $N \to \infty$. Typically the optimal order of perturbation theory is inversely proportional to the coupling constant. For quantum electrodynamics (QED), where the coupling constant is the fine structure constant $\alpha = e^2/\hbar c \approx \frac{1}{137}$, we lose the ability to calculate in a reasonable time long before we get to 137 loops, so practically speaking no problems arise from the lack of convergence. In quantum chromodynamics (QCD), however, the effective coupling constant is about two orders of magnitude larger, and perturbation theory is a much more subtle affair.