PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #3

(1) Consider a system of noninteracting spin trimers, each of which is described by the Hamiltonian

$$\hat{H} = -J\big(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\big) - \mu_0 \mathsf{H}\big(\sigma_1 + \sigma_2 + \sigma_3\big) \ .$$

The individual spin polarizations σ_i are two-state Ising variables, with $\sigma_i = \pm 1$.

(a) Find the single trimer partition function ζ.

(b) Find the magnetization per trimer $m = \mu_0 \, \langle \sigma_1 + \sigma_2 + \sigma_3 \rangle$.

(c) Suppose there are $N_\triangle$ trimers in a volume V. The magnetization density is $M = N_\triangle m/V$. Find the zero field susceptibility $\chi(T) = (\partial M/\partial \mathsf{H})_{\mathsf{H}=0}$.

(d) Find the entropy $S(T, \mathsf{H}, N_\triangle)$.

(e) Interpret your results for parts (b), (c), and (d) physically for the limits $J \to +\infty$, $J \to 0$, and $J \to -\infty$.

(2) The potential energy density for an isotropic elastic solid is given by

$$\begin{aligned} \mathcal{U}(\boldsymbol{x}) &= \mu \, \mathsf{Tr}\, \varepsilon^2 + \tfrac{1}{2}\lambda \, (\mathsf{Tr}\,\varepsilon)^2 \\ &= \mu \sum_{\alpha,\beta} \varepsilon^2_{\alpha\beta}(\boldsymbol{x}) + \tfrac{1}{2}\lambda \Big(\sum_\alpha \varepsilon_{\alpha\alpha}(\boldsymbol{x}) \Big)^2 \ , \end{aligned}$$

where μ and λ are the Lamé parameters and

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\bigg(\frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha} \bigg) \ ,$$

with $\boldsymbol{u}(\boldsymbol{x})$ the local displacement field. The Cartesian indices α and β run over x, y, z. The kinetic energy density is

$$\mathcal{T}(\boldsymbol{x}) = \tfrac{1}{2}\rho \, \dot{\boldsymbol{u}}^2(\boldsymbol{x}) \ .$$

(a) Assume periodic boundary conditions, and Fourier transform to wavevector space,

$$\begin{aligned} u^\alpha(\boldsymbol{x}, t) &= \tfrac{1}{\sqrt{V}} \sum_{\boldsymbol{k}} \hat{u}^\alpha_{\boldsymbol{k}}(t) \, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \\ \hat{u}^\alpha_{\boldsymbol{k}}(t) &= \tfrac{1}{\sqrt{V}} \int \! d^3\!x \; u^\alpha(\boldsymbol{x}, t) \, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \ . \end{aligned}$$

Write the Lagrangian $L = \int \! d^3\!x \, \big(\mathcal{T} - \mathcal{U}\big)$ in terms of the generalized coordinates $\hat{u}^\alpha_{\boldsymbol{k}}(t)$ and generalized velocities $\dot{\hat{u}}^\alpha_{\boldsymbol{k}}(t)$.

(b) Find the Hamiltonian H in terms of the generalized coordinates $\hat{u}^\alpha_{\boldsymbol{k}}(t)$ and generalized momenta $\hat{\pi}^\alpha_{\boldsymbol{k}}(t)$.

(c) Find the thermodynamic average $\langle \boldsymbol{u}(0) \cdot \boldsymbol{u}(\boldsymbol{x}) \rangle$.

(d) Suppose we add in a nonlocal interaction of the strain field of the form

$$\Delta U = \tfrac{1}{2} \int d^3x \int d^3x' \ \mathsf{Tr}\,\varepsilon(\boldsymbol{x}) \ \mathsf{Tr}\,\varepsilon(\boldsymbol{x}') \ v(\boldsymbol{x} - \boldsymbol{x}') \ .$$

Repeat parts (b) and (c).