Practice Midterm

Mid-Term Exam
203B — 2009

Your Name

All problems worth 10 points.

1. The frequency-dependent dielectric for a plasma is approximately $\varepsilon(\omega) \simeq 1 - \omega_p^2/\omega^2$, where $\omega_p$ is the plasma frequency. For the ionosphere, the maximum value of this frequency is approximately $\omega_p \simeq$ ~~10 MHz.~~ $2 \cdot 10^7\,\text{sec}^{-1}$ It is well known that long wave length radio waves ($\lambda \gtrsim 1000$m) follow the curvature of the earth, bouncing back and forth between the ionosphere and the surface of the earth, easily having a continental range. On the other hand, TV waves ($\lambda \sim 1$m) don't follow the curvature of the earth; TV reception is line of sight from antenna to receiver. Give a short explanation of these observations.

The waves can pass through the ionosphere only if $\omega > \omega_p$

for $\lambda = 1000\,\text{m}$ $\quad \omega = \dfrac{2\pi c}{\lambda} = \dfrac{2\pi \cdot 3 \cdot 10^8}{10^3}\,\text{sec}^{-1}$

$$\omega = 2 \cdot 10^6\,\text{sec}^{-1} < \omega_p$$

So long wavelength waves reflected from ionosphere

for $\lambda = 1\,\text{m}$ $\quad \omega = \dfrac{2\pi c}{\lambda} = \dfrac{2\pi \cdot 3 \cdot 10^8}{1\,\text{m}}\,\text{sec}^{-1}$

$$\therefore \omega \simeq 2 \cdot 10^9\,\text{sec}^{-1} > \omega_p$$

T.V waves propagate through ionosphere

2. A long copper bar of rectangular cross section (see figure) is initially immersed in a uniform axial magnetic field $\mathbf{B} = \hat{z}B_0$. At $t = 0$, the external support for the magnetic field is suddenly switched off, and the field in the copper bar begins to decay slowly. Find the magnetic field in the bar for $t > 0$ [i.e., find $B_z(x,y,t)$].

$$\nabla^2 B_z = \frac{4\pi\sigma}{c^2}\frac{\partial B_z}{\partial t}, \quad B_z = 0 \text{ at } x = 0, a; \; y = 0, b$$

$$B_z = X(x)Y(y)T(t)$$

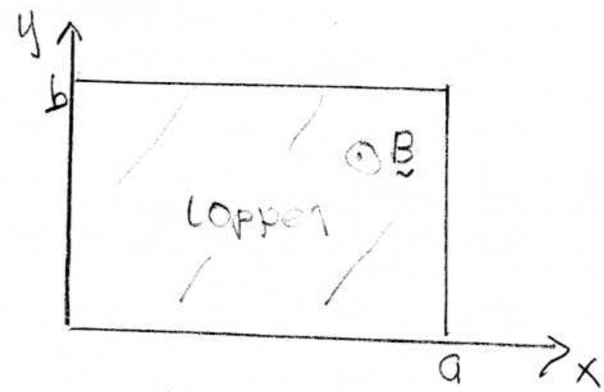
$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{4\pi\sigma}{c^2}\frac{\partial T}{\partial t}\frac{1}{T}$$

$$X = \sin\frac{n\pi x}{a} \qquad Y = \sin\frac{n\pi}{b}y$$

$$T = e^{-\gamma_{nm}t}, \quad \gamma_{nm} = \frac{c^2}{4\pi\sigma}\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]$$

$$B_z(x,y,t) = \sum_{n,m} A_{nm}\sin\frac{n\pi}{a}x\,\sin\frac{m\pi}{b}y\,e^{-\gamma_{nm}t}$$

$$A_{nm}\frac{a}{2}\frac{b}{2} = \int_0^a dx\int_0^b dy\,\sin\frac{n\pi}{a}x\,\sin\frac{n\pi}{b}y\,B_0$$



3. According to the Lorentz-Drude model for a metal, the current density and the electric field are related by the equation

$$\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t) + \gamma\mathbf{J}(\mathbf{r},t) = \frac{ne^2}{m}\mathbf{E}(\mathbf{r},t),$$

where $n$ is the density of conduction electrons and $\gamma$ is the collisional relaxation rate for the current (in the absence of an electric field).

a. Obtain ~~a~~ an ordinary differential equation that governs the temporal dependence of the charge density, $\rho(\mathbf{r},t)$, at each point $\underline{r}$ inside the conductor.

$$0 = \frac{\partial\rho}{\partial t} + \nabla\cdot\underline{J} \qquad \underline{\nabla}\cdot\underline{E} = 4\pi\rho$$

$$-\frac{\partial^2\rho}{\partial t^2} - \gamma\frac{\partial\rho}{\partial t} = \frac{4\pi n e^2}{m}\rho \qquad \frac{4\pi n e^2}{m} = \omega_p^2$$

$$\frac{\partial^2\rho}{\partial t^2} + \gamma\frac{\partial\rho}{\partial t} + \omega_p^2\rho = 0$$

b. Given that $\rho(\mathbf{r},t=0) = \rho_0(\mathbf{r})$ and $\mathbf{J}(\mathbf{r},t=0) = 0$, find $\rho(\mathbf{r},t)$ for $t > 0$. You may assume that $\gamma^2 << ne^2/m$. Also, give an estimate of the time for $\rho(\mathbf{r},t)$ to decay for the case where the conductor is copper at room temperature.

$$\left.\frac{\partial\rho}{\partial t}\right|_{t=0} = -\nabla\cdot\underline{J}\Big|_{t=0} = 0, \qquad \rho(\underline{r},0) = \rho_0(\underline{r})$$

Let $\rho(\underline{r},t) \sim \rho(\underline{r})e^{st}$

$$s^2 + \gamma s + \omega_p^2 = 0, \qquad s = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_p^2}}{2} \simeq -\frac{\gamma}{2} \pm i\omega_p$$

$$\rho(\underline{r},t) = \rho_0(\underline{r})e^{-\frac{\gamma}{2}t}\cos\omega_p t + \frac{\gamma}{2\omega_p}\rho_0(\underline{r})e^{-\frac{\gamma}{2}t}\sin\omega_p t$$

$\gamma/2 \simeq 10^{13}$ sec$^{-1}$

4. A plane electromagnetic wave propagates in the +x-direction and has intensity (time-average power/area) $\langle S_z \rangle = I_0$. The wave reflects normally from a plane conductor (mirror) that is moving toward the wave with velocity $-V\hat{x}$. What is the intensity of the reflected wave (in the laboratory frame)?

In moving frame

$$\bar{E}_\perp = \gamma\left[E_\perp + \frac{V \times B}{c}\right] = \gamma\left(1+\frac{V}{c}\right)E_\perp\,\hat{y}$$

after reflection

$$\bar{E}'_\perp = -\bar{E}_\perp = -\gamma\left(1+\frac{V}{c}\right)E_\perp\,\hat{y}\,, \quad \bar{B}' = -\gamma\left(1+\frac{V}{c}\right)E_\perp\,\hat{x}$$

back to lab frame

$$E'_\perp = \gamma\left[\bar{E}'_\perp - \frac{V}{c}\times\bar{B}'_\perp\right] = \gamma\left[1+\frac{V}{c}\right]\bar{E}'_\perp$$

$$|E'_\perp| = \gamma^2\left(1+\frac{V}{c}\right)^2|E_y| = \frac{\left(1+\frac{V}{c}\right)^{\cancel{2}}}{\left(1-\frac{V}{c}\right)\cancel{\left(1+\frac{V}{c}\right)}}|E_y|$$

$$\langle S_z\rangle = \left(\frac{1+V/c}{1-\frac{V}{c}}\right)^2 I_0$$

after reflection