5/6/03 start

Reflection coefficient

$$\langle \underset{\sim}{S} \rangle = \frac{1}{2} \mathrm{Re} \frac{\underset{\sim}{E} \times \underset{\sim}{H}^* c}{4\pi}$$

$$R = \frac{|\langle S_y'' \rangle|}{|\langle S_y \rangle|} = \frac{\left(\sqrt{\cancel{\epsilon_1}}\ \cancel{\cos\theta''}\ |E''|\right)^2}{\sqrt{\cancel{\epsilon_1}}\ \cancel{\cos\theta}\ |E|^2}, \qquad \frac{\mu_1}{\mu_2} \simeq 1$$

$$\left\{ \begin{array}{l} i\underset{\sim}{k} \times \underset{\sim}{H} = -\frac{i\omega}{c} \epsilon \underset{\sim}{E} \\ k^2 c^2 = \omega^2 \epsilon \mu \end{array} \right.$$

normal incidence

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

Polarization ⊥ to plane of incidence

$$R_\perp = \left(\frac{n_1 \cos\theta - n_2 \cos\theta'}{n_1 \cos\theta + n_2 \cos\theta'} \right)^2$$

$$R_\perp = \left(\frac{\sin\theta' \cos\theta - \overbrace{\frac{n_2}{n_1} \sin\theta'}^{\sin\theta} \cos\theta'}{\sin\theta' \cos\theta + \underbrace{\frac{n_2}{n_1} \sin\theta'}_{\sin\theta} \cos\theta'} \right)^2$$

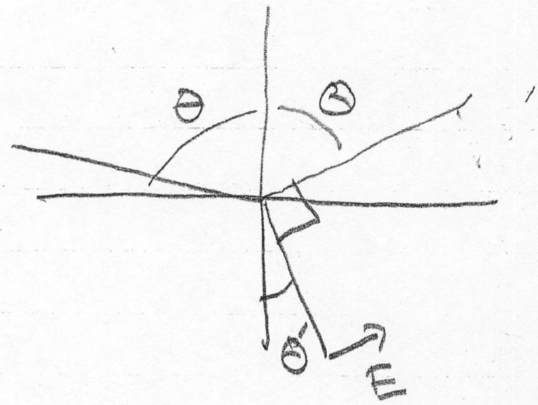
$$R_\perp = \left[\frac{\sin(\theta' - \theta)}{\sin(\theta + \theta')} \right]^2$$

Homework

(18) Show that for polarization parallel to plane of incidence

$$R_{11} = \left(\frac{\tan(\theta-\theta')}{\tan(\theta+\theta')}\right)^2$$

<u>Brewster angle</u>



if $\theta + \theta' = \pi/2$

$\tan(\theta+\theta') = \infty$

$\therefore \; R_{11} \to 0$

$n_1 \sin\theta = n_2 \sin\theta' = n_2 \sin(\pi/2 - \theta) = n_2 \cos\theta$

$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$

note that $R_{\perp} \neq 0$

discuss from point of view of dipole radiators

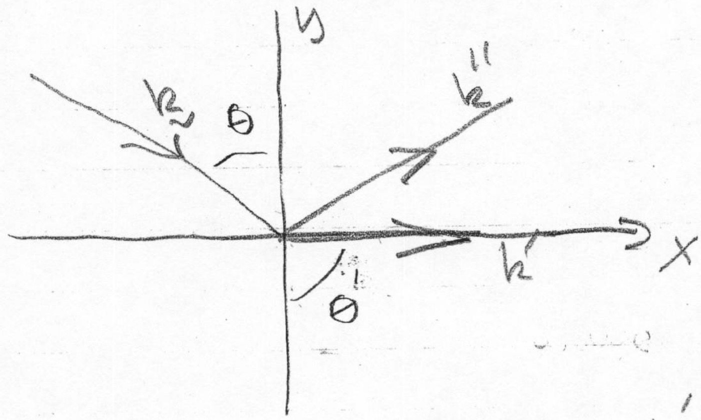
discuss glare reduction by polarized sun glasses total internal reflection

let $n_1 > n_2$

for $\frac{n_1}{n_2}\sin\theta > 1$

$$1 < \frac{n_1}{n_2}\sin\theta = \sin\theta'$$

no solns for real θ'



$\frac{n_1}{n_2}\sin\theta = 1$

matching requires $k_x = k_x'$, $\omega = \omega'$

but $(k_y^2 + k_x^2)\frac{c^2}{n_1^2} = \omega^2 = \omega'^2 = \frac{c^2}{n_2^2}(k_x'^2 + k_y'^2)$

$$1 > \frac{1}{\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta} = \frac{k^2}{k_x^2}\frac{n_2^2}{n_1^2} = \frac{k_x'^2}{k_x^2} + \frac{k_y'^2}{k_x^2} = 1 + \frac{k_y'^2}{k_x^2}$$

is negative

$k_y'^2 < 0$

$k_y' = -i|k_y'|$

$e^{ik_y'y} = e^{+|k_y'|y}$ decreases as y becomes more negative

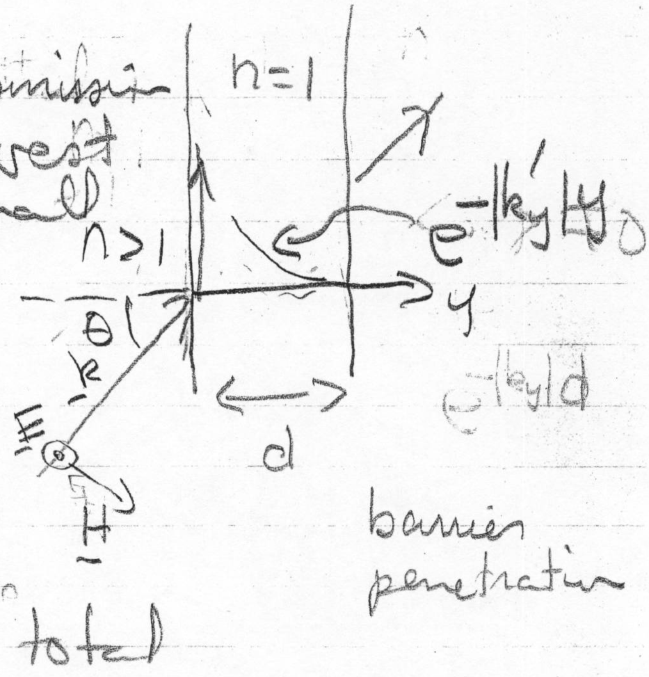
$\langle S_y \rangle = 0$ if medium ② extends to $y = -\infty$

Homework

(19) (2) ~~Calculate the transmission coefficient to lowest order in the small parameter,~~

(25) $e^{-|k_y'|d}$,

~~assuming that~~ θ is ~~greater than~~ critical angle for total internal reflection.



Stent

5/8/03

Simple model for $\epsilon(\omega)$

assumes low density (mention Clausius-Mossotti eq.)

Approximate each electron in atom by classical oscillator

$$m[\ddot{\underline{r}} + \gamma\dot{\underline{r}} + \omega_0^2 \underline{r}] = \mathrm{Re}(-e\underline{E}\, e^{i\underline{k}\cdot\underline{r} - i\omega t})$$

$$|\underline{k}\cdot\Delta\underline{r}| \ll 1$$

$$\Delta\underline{r} = \mathrm{Re}\,\frac{-\frac{e}{m}\underline{E}\, e^{i\underline{k}\cdot\underline{r} - i\omega t}}{-\omega^2 + \omega_0^2 - i\omega\gamma}$$

$$\underline{p} = -e\Delta\underline{r}$$

density atoms

$$\epsilon(\omega) = 1 + \frac{4\pi n e^2}{m}\sum_j \frac{f_j}{-\omega^2 + \omega_j^2 - i\omega\gamma_j}$$

electrons in atom

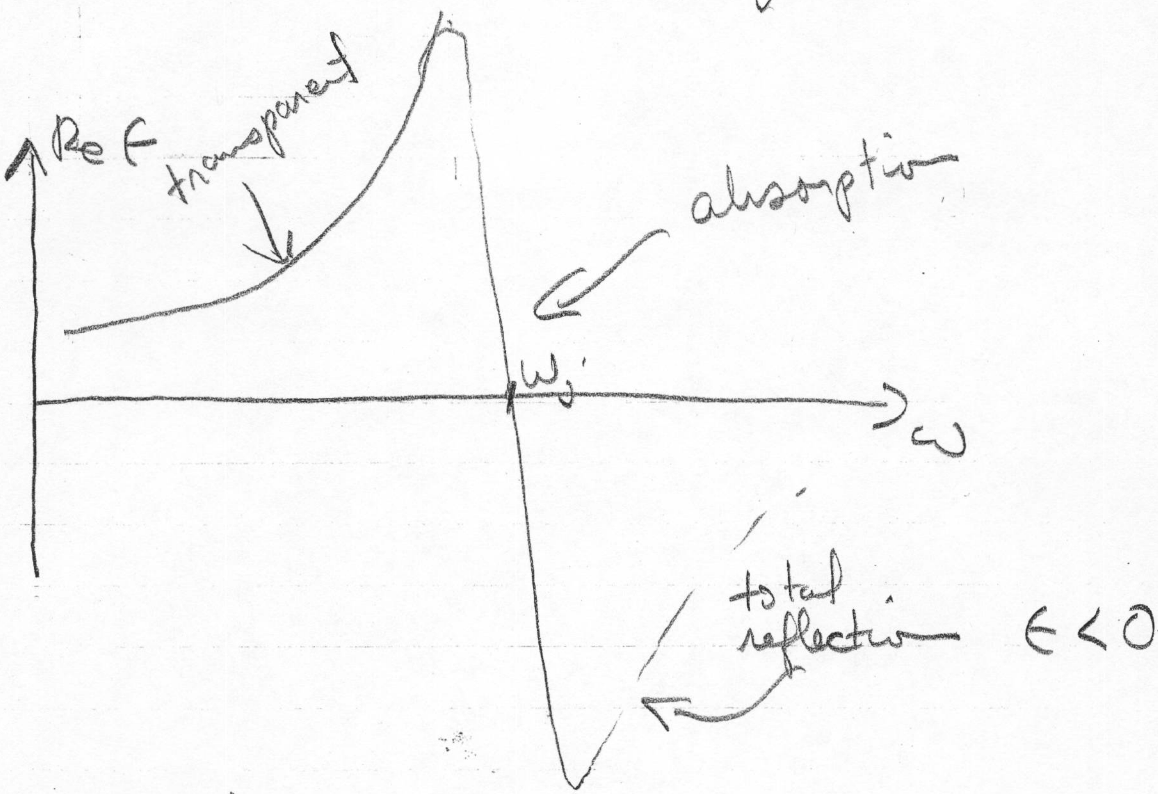
$$\sum_j f_j = Z$$

oscillator strengths

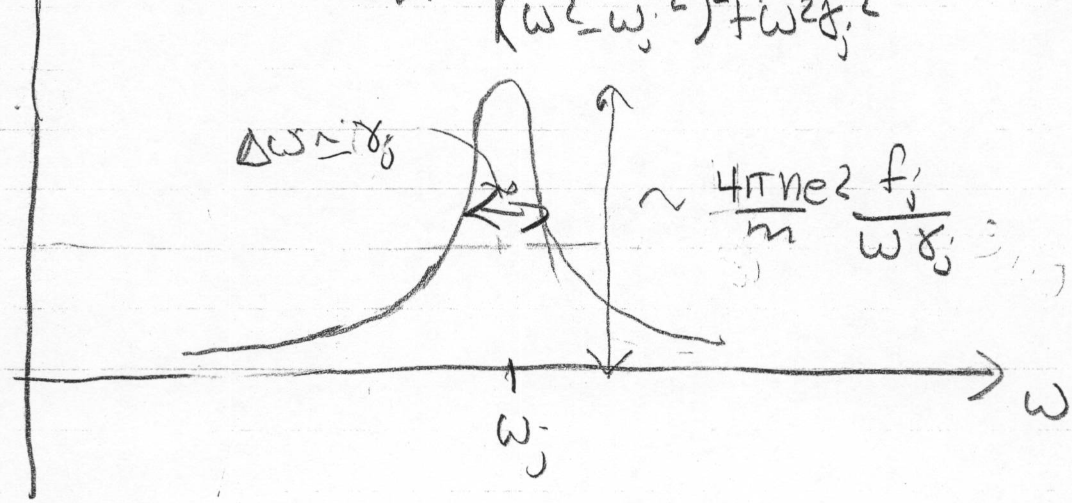
quantum mechanical interpretations f_i, ω_i, γ_i

near a resonance

$$\mathrm{Re}\,\epsilon(\omega) \simeq 1 + \frac{4\pi n e^2}{m} \frac{f_j(-\omega^2+\omega_j^2)}{(-\omega^2+\omega_j^2)^2+\gamma_j^2\omega^2}$$



$$\mathrm{Im}(\epsilon) \simeq \frac{4\pi n e^2}{m} \frac{f_j \gamma_j \omega}{(\omega^2-\omega_j^2)^2+\omega^2\gamma_j^2}$$



Insulators versus conductors

insulator $\quad \omega_j > 0$ for all j.

(no free electrons)

conductor $\quad \omega_{j=1} = 0$

(some free or conduction band electrons)

low freq. limit for conductor

$$\epsilon(\omega) = \underbrace{1 + \frac{4\pi n e^2}{m} \sum_{j=2}^{Z} \frac{f_j}{\omega_j^2}}_{\epsilon_0} - \underbrace{\frac{4\pi n e^2 f_1}{m[\cancel{\omega^2} + i\omega\gamma]}}_{\substack{\text{neglect} \\ \text{for } |\omega| << \gamma}}$$

from

$$\nabla \times \underset{\sim}{H} = \frac{4\pi}{c}\sigma \underset{\sim}{E} + \frac{\epsilon_0}{c}\frac{\partial \underset{\sim}{E}}{\partial t} = \frac{\epsilon}{c}\frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\frac{4\pi}{c}\sigma - \frac{i\omega\epsilon_0}{c} = -\frac{i\omega}{c}\epsilon(\omega)$$

slant

$$\therefore \quad \sigma = \frac{-i\omega}{4\pi}\left(\frac{4\pi n e^2 f_1}{m\, i\omega\gamma_1}\right) = \frac{n e^2 f_1}{m\gamma_1}$$

$f_1 n \simeq$ density of free electrons

$\gamma = \frac{1}{\tau}$ ← momentum transfer time for collisions with ions (i.e., phonons)

$$-m\frac{1}{\tau}\dot{\underset{\sim}{r}} = e\,\mathrm{Re}\,\underset{\sim}{E}\,e^{-i\omega t}, \quad J = nfe\,\dot{\underset{\sim}{r}}$$

$$\sigma = \frac{(nf)e^2\tau}{m}$$

for copper $\sigma \simeq 5\times10^{17}\ \mathrm{sec}^{-1}$

$nf \simeq 8\times10^{22}\ \mathrm{cm}^{-3}$

$$\tau \simeq \frac{m\sigma}{(nf)e^2} \simeq \frac{(10^{-27}\,\mathrm{gm})(5\times10^{17}\,\mathrm{sec}^{-1})}{(8\times10^{22}\,\mathrm{cm}^{-3})(25\times10^{-20}\,\mathrm{statcoul}^2)}$$

$\tau \simeq 3\cdot10^{-14}$ sec

note that $\ell_{mfp} \simeq \left(5\times10^{7}\frac{\mathrm{cm}}{\mathrm{sec}}\right) 3\cdot10^{-14}\,\mathrm{sec} \simeq 1.5\cdot10^{-6}\,\mathrm{cm} \simeq 10^{2}\,Å$ ← 1eV

scattering is weak because of lattice

note that low freq. approx holds for

$$\omega \ll \gamma = \frac{1}{\tau} \simeq 3\times 10^{13}\ \mathrm{sec}^{-1}$$

<u>high freq limit for conductors</u>

(plasma limit)

$$\omega \gg \gamma$$

(e.g., optical)

$$\epsilon(\omega) \simeq \epsilon_0(\omega) - \frac{4\pi e^2 n f}{m\omega^2}$$

ω_p^2

for $\gamma \ll \omega \lesssim \omega_p$ (optical)

k_I^{-1} is collisionless skin depth

$$k^2c^2 = \epsilon\,\omega^2 \simeq -\omega_p^2$$

$$k_R \simeq 0, \quad k_I \simeq \frac{\omega_p}{c}$$

wave is reflected from surface

mention reflection at ionosphere

for $\omega \gg \omega_p$

$$k^2c^2 \simeq \omega^2\epsilon_0 > 0$$

metals transparent in ultraviolet

note that all substances look like plasma for sufficiently high ω (i.e. $\omega \gg \omega_j$ for all j)

$$\epsilon(\omega) \simeq 1 - \frac{4\pi n e^2}{m\omega^2}\sum_j^{Z} f_j = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\omega_p^2 = \frac{4\pi e^2 n Z}{m}$$

waves

Homework

(20) J 7, 12

Incorrect sol'n

$$\frac{\partial\rho}{\partial t} + \nabla\cdot J = 0 \quad , \quad J = \sigma E$$

$$\frac{\partial\rho}{\partial t} + \sigma \underbrace{\nabla\cdot E}_{4\pi\rho} = 0 \quad , \quad \rho(r,t) = \rho(r,0)\,e^{-4\pi\sigma t}$$

why incorrect

$$\left|\frac{1}{\rho}\frac{\partial\rho}{\partial t}\right| = 4\pi\sigma \simeq 10.5\times10^{17}\,\text{sec}^{-1} \gg \underset{3\times10^{13}\,\text{sec}^{-1}}{\underset{\parallel}{\gamma}}$$

Quasi-static magnetic field near a conductor

L+L Vol 8 Chapter 7

Assume that

$$L << c/\omega$$

↑ scale length of conductor

$\therefore$ field outside conductor can be treated as quasi-static

$$\underset{\sim}{\nabla} \cdot \underline{B} = 0 \qquad \underset{\sim}{\nabla} \times \underline{H} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \leftarrow \text{small}$$

Inside conductor, assume that

$$\omega << \gamma \quad \text{and} \quad \omega << \sigma$$

$$\therefore \underset{\sim}{\nabla} \times \underset{\sim}{H} = \frac{4\pi\sigma E}{c} + \frac{\epsilon_0}{c}\frac{\partial E}{\partial t} \simeq \frac{4\pi\sigma}{c}\underset{\sim}{E}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

Also, let $\underset{\sim}{B} = \mu \underset{\sim}{H}$ in conductor

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi\sigma\mu}{c} \underset{\sim}{E}$$

$$\underbrace{\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \times \underset{\sim}{B}}_{\underset{\sim}{\nabla}\,\underset{\sim}{\nabla} \cdot \underset{\sim}{B} - \nabla^2 \underset{\sim}{B}} = \frac{4\pi\sigma\mu}{c} \underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{4\pi\sigma\mu}{c^2} \frac{\partial \underset{\sim}{B}}{\partial t}$$

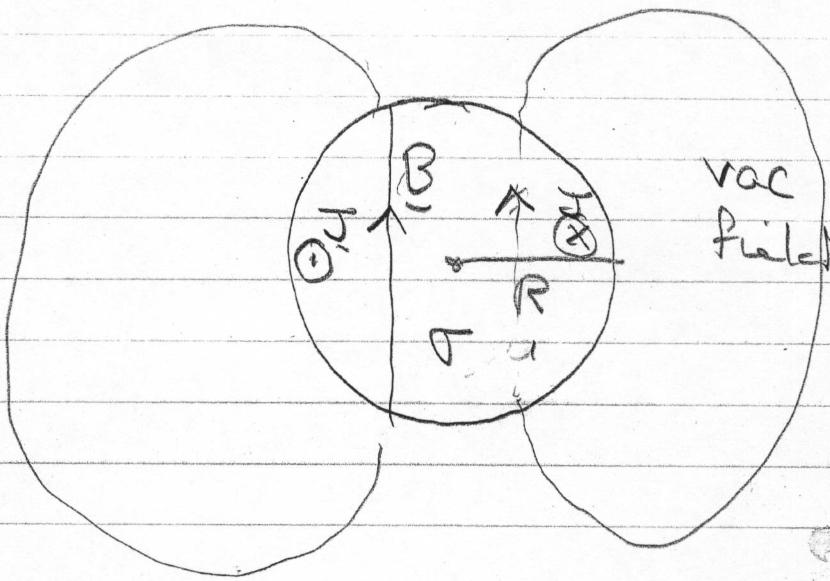
$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

$$\therefore \quad \nabla^2 \underset{\sim}{B} = \frac{4\pi\sigma\mu}{c^2} \frac{\partial \underset{\sim}{B}}{\partial t}$$

First order in time, so describes irreversible process. (magnetic field energy is dissipated through Joule heating)

Vector version of diffusion or heat conduction equation

Suppose a copper sphere is suddenly removed from external magnetic field B_0. The field persists because currents are induced as the field tries to escape



Discuss current production and decay physically

Estimate time of decay

$$\underbrace{\nabla^2 \underset{\sim}{B}}_{\sim \frac{1}{R^2}\cancel{\underset{\sim}{B}}} = \frac{4\pi\sigma\mu}{c^2}\underbrace{\frac{\partial \underset{\sim}{B}}{\partial t}}_{\sim \cancel{B}/\tau}$$

$$\therefore \quad \tau \simeq R^2 \frac{4\pi\sigma\mu}{c^2}$$

for copper sphere of $R = 10$ cm

$$\tau \simeq \frac{(100\,\mathrm{cm}^2)(4\pi)(5\times10^{17}\,\mathrm{sec}^{-1})(1)}{(3\times10^{10}\,\mathrm{cm/sec})^2} \simeq .5\ \mathrm{sec}$$

start

~~For magnetic field in molten iron core of earth~~

$R_c \simeq 4\times10^8$ cm , $\mu = 1$ above Curie Temp.

$\sigma \simeq 10^{16}\,\mathrm{sec}^{-1}$

$$\tau \simeq \frac{(16\times10^{16})\,4\pi\,(10^{16})}{(3\times10^{10})^2}\,\mathrm{sec} \simeq 2\times10^{13}\,\mathrm{sec}$$

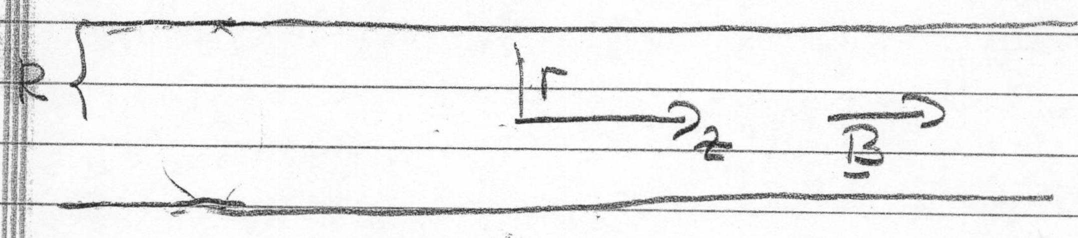
$$\simeq 10^6\ \mathrm{years}$$

~~discuss dynamo effect~~
~~diffusion time for mantle~~

start

Homework (21) (from departmental exam

A long copper cylinder of radius R resides in a uniform axial magnetic field $\underline{B}_0 = \hat{z} B_0$. When the external support for the field is suddenly switched off, the field in the copper decays slowly. Determine $B_z(r,t)$ inside the copper.



Hint: at $t=0$, $B_z(r, t=0) = B_0$

at $r=R$ $\quad B_z(R,t) = 0$

solve

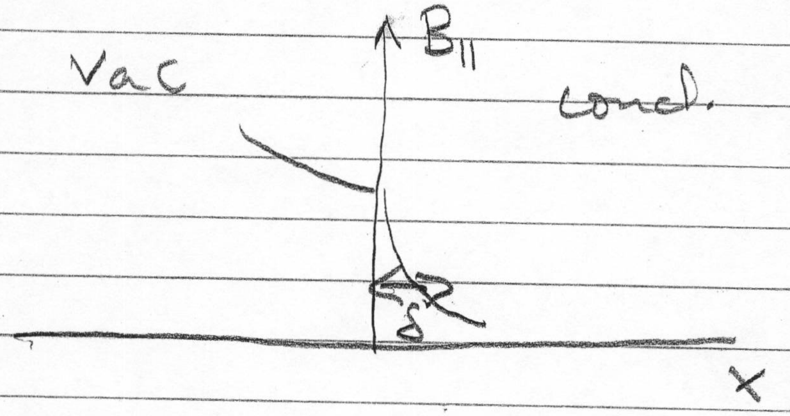
$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} B_z = \nabla^2 B_z = \frac{4\pi\sigma}{c^2}\frac{\partial B_z}{\partial t}$$

using eigenfunction expansion

Skin depth

Consider a conductor in the region of an external field $\underset{\sim}{B} e^{-i\omega t}$

The field penetrates only a finite distance into the conductor



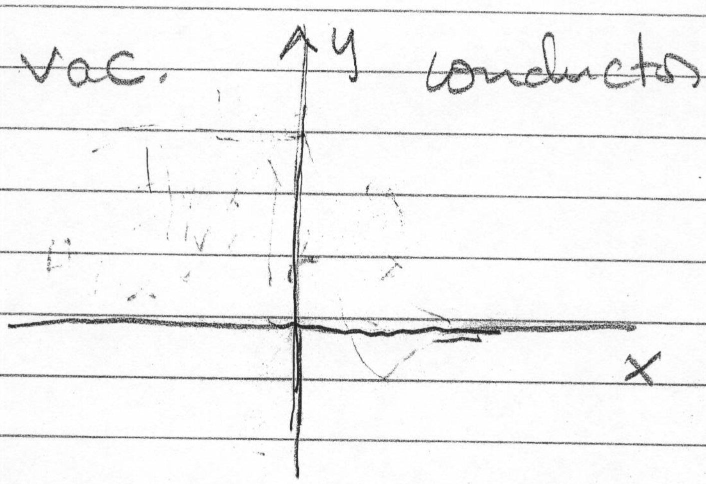
The field diffuses in for a time $T \simeq \omega^{-1}$ and then reverses direction and starts over

$$\delta^2 \sim R^2 \qquad T \sim \omega^{-1}$$

$$\delta \sim \sqrt{\frac{c^2}{\omega \sigma \mu}}$$

When $\delta <<$ Radius of curvature of conductor,

can treat interface as plane



In zeroth approx, let $\delta \to 0$ and solve for vacuum fields assuming that conductor is perfect.

In first approximation, solve for field in transition region using vacuum soln as b.c. at interface.

let $H_{vac}(x=0^-,t) = H_0 e^{-i\omega t}\hat{y}$

from order zero solns.

$H_{cond}(x,t) = H(x) e^{-i\omega t}\hat{y}$

where $H_0 = H(0+)$ current extends into conductor distance δ

$$\nabla^2 H = -\frac{i\omega 4\pi\sigma\mu}{c^2} H$$

$\frac{\partial^2}{\partial x^2}$

draw in figure

$$H(x) \sim e^{kx}$$

picture

$$k^2 = -\frac{i\omega 4\pi\sigma\mu}{c^2}, \quad k = \pm\sqrt{\frac{4\pi\sigma\mu\omega}{c^2}}\frac{(1-i)}{\sqrt{2}}$$

$$H(x) = H_0 \exp\left[-\underbrace{\sqrt{\frac{2\pi\sigma\mu\omega}{c^2}}}_{\equiv 1/\delta}(1-i)\,x\right]$$

skin depth (resistive skin depth)

electric field

✓ $$\frac{4\pi\sigma}{c}\underset{\sim}{E} \simeq \underset{\sim}{\nabla}\times\underset{\sim}{H} = \hat{z}\frac{(i-1)}{\delta}H(x)$$

✓ $$\underset{\sim}{E} = \hat{z}(i-1)\underbrace{\frac{c}{4\pi\sigma\delta}}_{\sim\sqrt{\frac{\mu\omega}{\sigma}} \ll 1}H(x)$$

draw H_y, E_z, E_x in figure

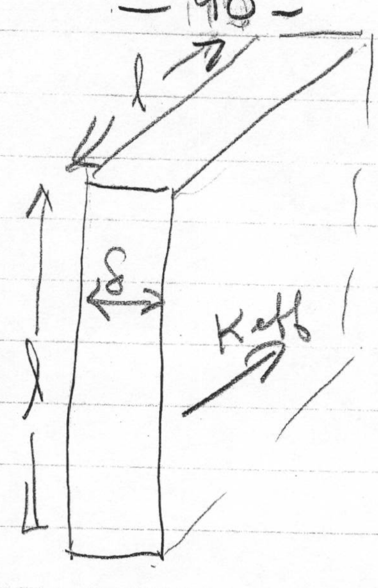
5/15/03 ✓ $$\frac{\langle\text{power dissipated}\rangle}{\text{area}} = \int_0^\infty \sigma\langle E^2\rangle dx$$

$$\text{''} = \frac{1}{2}\sigma|E_z(0)|^2\underbrace{\int_0^\infty dx\, e^{-2x/\delta}}_{\delta/2}$$

✓ $$= \frac{\sigma\delta}{4}\,2\left(\frac{c}{4\pi\sigma\delta}\right)^2|H_0|^2 = \underbrace{\frac{1}{2\sigma\delta}}_{R_{eff}}\left(\frac{c}{4\pi}H_0\right)^2$$

$$\sim\sqrt{\frac{\omega}{\sigma}}$$

start



$$\frac{\langle I^2 R\rangle}{l^2} = \frac{1}{2l^2}\left(l K_{eff}\right)^2\left(\frac{l}{\sigma l \delta}\right) \simeq \frac{\left(K_{eff}\right)^2}{2\sigma\delta}$$

Homework

(22) Calculate power/area dissipated using Poynting vector

<u>numerical examples</u>

for copper at room temp ($\sigma \simeq 5\times10^{17}\,\mathrm{sec}^{-1}$)

$\delta \simeq$ cm for $\nu = 60\,\mathrm{Hz}$

$\delta \simeq 10^{-3}$ cm for $\nu = 10^{8}\,\mathrm{Hz}$

Wave guides and resonant cavities

J. Chapter 8, L+L vol 8 Chapter 10

Can't use static approximation for fields in vacuum region.

$$\frac{c}{\omega} \sim L$$

Can still use low freq. approx. in conductor

$$\omega \ll \gamma, \sigma$$

Plane interface approx. valid for skin depth

$$\delta \ll L$$

numerical example

$$L \simeq 1\,\text{cm} \Rightarrow \omega \simeq 10^{10}\,\text{sec}^{-1}$$

for copper at room temp

$$\gamma \simeq 3 \cdot 10^{13}\,\text{sec}^{-1} \qquad \sigma \simeq 5 \times 10^{17}\,\text{sec}^{-1}$$

$$\delta \simeq 10^{-4}\,\text{cm}$$

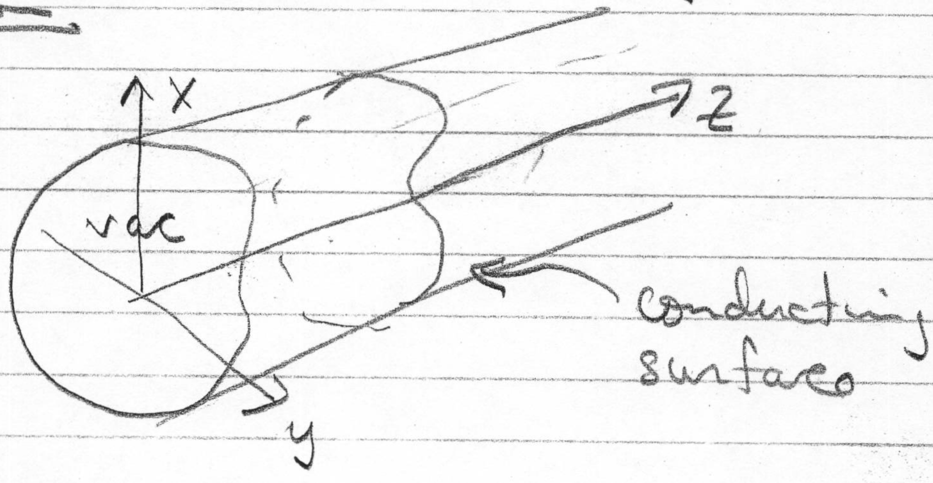
To solve for field in vacuum region use b.c. for perfect conductor

$$\hat{n} \cdot \underset{\sim}{E} = 4\pi \Sigma$$

$$\hat{n} \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{K}$$

$$* \quad \hat{n} \cdot \underset{\sim}{B} = 0 \quad , \quad \hat{n} \times \underset{\sim}{E} = 0$$

<u>Hollow guide of arbitrary cross section</u>



$$\underset{\sim}{E}, \underset{\sim}{B} \sim e^{-i\omega t}$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = \frac{i\omega}{c} \underset{\sim}{B} \quad , \quad \underset{\sim}{\nabla} \times \underset{\sim}{B} = -\frac{i\omega}{c} \underset{\sim}{E}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

$$\therefore \left[\nabla^2 + \frac{\omega^2}{c^2}\right]\left[\underset{\sim}{E}, \underset{\sim}{B}\right] = 0$$

$$\underset{\sim}{E}(x,y,z) = \underline{E}(x,y)e^{ikz}$$

$$\underline{B}(x,y,z) = \underline{B}(x,y)e^{ikz}$$

$$\left[\nabla_t^2 + \frac{\omega^2}{c^2} - k^2\right]\left[\underline{E}(x,y), \underline{B}(x,y)\right] = 0$$

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Let

$$\underset{\sim}{E} = \underset{\sim}{E}_z + \underset{\sim}{E}_t \quad , \quad \underset{\sim}{B} = \underset{\sim}{B}_z + \underset{\sim}{B}_t$$

Solutions are usually divided into three categories:

transverse magnetic (TM)

$$B_z = 0 \quad \text{everywhere}$$

transverse electric (TE)

$$E_z = 0 \quad \text{everywhere}$$

transverse electromagnetic (TEM)

$$E_z, B_z = 0 \quad \text{everywhere}$$

<u>case 1</u> (TM)

$$\frac{\partial E_z}{\partial y} - ikE_y = \frac{i\omega}{c} B_x$$

$$-\frac{\partial E_z}{\partial x} + ikE_x = \frac{i\omega}{c} B_y$$

$$-ikB_y = -\frac{i\omega}{c} E_x \quad , \quad ikB_x = -\frac{i\omega}{c} E_y$$

$\therefore$

$$E_x = \frac{c}{\omega} k B_y = \frac{c}{\omega} k \left[\frac{ic}{\omega} \frac{\partial E_z}{\partial x} + \frac{ck}{\omega} E_x \right]$$

save

$$\therefore \quad E_x = \frac{ik\, \partial E_z/\partial x}{\frac{\omega^2}{c^2} - k^2}, \qquad B_y = \frac{\frac{i\omega}{c} \frac{\partial E_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2}$$

$$E_y = \frac{ik\, \partial E_z/\partial y}{\frac{\omega^2}{c^2} - k^2}, \qquad B_x = \frac{-\frac{i\omega}{c} \partial E_z/\partial y}{\frac{\omega^2}{c^2} - k^2}$$

boundary conditions

$$0 = \hat{n} \cdot \underset{\sim}{B} = n_x B_x + n_y B_y \propto \underbrace{\left(-\frac{\partial E_z}{\partial y} n_x + n_y \frac{\partial E_z}{\partial x} \right)}_{\hat{z} \cdot \left[-\hat{n} \times \underset{\sim}{\nabla} E_z \right]}$$

$$\underbrace{\hat{z} \cdot \left[-\hat{n} \times \underset{\sim}{\nabla} E_z \right]}_{\underbrace{\hat{n} \times \hat{z} \cdot}_{\frac{d}{dl}} \underset{\sim}{\nabla} E_z}$$

along circumference $\rightarrow \frac{d}{dl}$

$\propto \quad n_x \frac{\partial E_z}{\partial y} - n_y \frac{\partial E_z}{\partial x} \propto \frac{dE_z}{dl}$

$(n_x E_y - n_y E_x)\hat{z}$

$$0 = \hat{n} \times \underset{\sim}{E} = \hat{n} \times \hat{z} E_z + \hat{n} \times \underset{\sim}{E_t}$$

Both b.c. satisfied if $E_z = 0$ on boundary

$\therefore$ E_z satisfies eigenvalue problem

$$\left[\nabla_t^2 + \frac{\omega^2}{c^2} - k^2\right] E_z(x,y) = 0$$

5/20/03 $\quad E_z(x,y) = 0$ on boundary

<u>case 2</u>

Homework

(23) Show that for TE waves

$$B_x = \frac{ik\, \partial B_z/\partial x}{\omega^2/c^2 - k^2}, \quad B_y = \frac{ik\, \partial B_z/\partial y}{\frac{\omega^2}{c^2} - k^2}$$

$$E_x = \frac{i\omega/c\ \partial B_z/\partial y}{\frac{\omega^2}{c^2} - k^2} \qquad E_y = \frac{-\frac{i\omega}{c}\frac{\partial B_z}{\partial x}}{\frac{\omega^2}{c^2} - k^2}$$

(continued next page)

$$\left[\nabla_t^2 + \left(\frac{\omega^2}{c^2} - k^2\right)\right] B_z(x,y) = 0$$

$$\frac{\partial B_z}{\partial n} = 0 \quad \text{on boundary}$$

<u>case 3</u> TEM mode

$\frac{\omega^2}{c^2}$ must equal k^2 so that

$B_z, E_z = 0$ but $\underset{\sim}{B}_t, \underline{E}_t \neq 0$

$$\left[\nabla_t^2 + \underbrace{\left(\frac{\omega^2}{c^2} - k^2\right)}_{=0}\right](\underset{\sim}{E}_t, \underset{\sim}{H}_t) = 0$$

1 <u>st</u> soln

$$\underset{\sim}{E}_t = \underset{\sim}{\nabla}_t \varphi(x,y)$$

where $\nabla_t^2 \varphi = 0$, $\varphi =$ const on boundary

$$\therefore \hat{n} \times \underset{\sim}{E} = \hat{n} \times \underset{\sim}{\nabla}_t \varphi = 0$$

on boundary

$$\frac{i\omega}{c}\,\underset{\sim}{B}_t = \underset{\sim}{\nabla}\times\left[\underset{\sim}{\nabla}_t \phi\, e^{ikz}\right] = ik\,\hat{z}\times\underset{\sim}{\nabla}_t\phi$$

$$\underset{\sim}{B}_t = \frac{ck}{\omega}\,\hat{z}\times\underset{\sim}{\nabla}_t\phi \quad , \quad \therefore\ \underset{\sim}{B}_t\cdot\hat{n} = 0$$

2nd soln

$$\underset{\sim}{B}_t = \underset{\sim}{\nabla}_t\phi \qquad \text{where}$$

$$\nabla_t^2\phi = 0 \quad , \quad \frac{\partial\phi}{\partial n} = 0 \ \text{ on boundary}$$

$\therefore\ B\cdot\hat{n} = 0$ on boundary

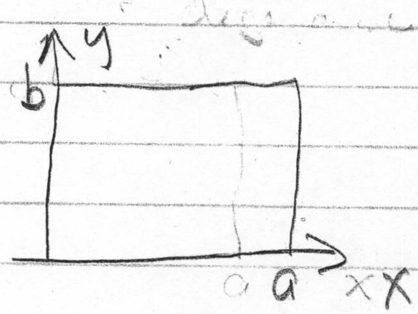
$$-\frac{i\omega}{c}\,\underset{\sim}{E}_t = \underset{\sim}{\nabla}\times\underset{\sim}{\nabla}_t\phi\, e^{ikz} = ik\,\hat{z}\times\underset{\sim}{\nabla}_t\phi$$

$$\therefore\ \hat{n}\times\underset{\sim}{E}_t = 0 \ \text{ on boundary}$$

note lack of 2nd soln in connected regions

start

<u>Rectangular wave guides</u>



<u>TM modes</u>

$$\left[\nabla_t^2 + \left(\frac{\omega^2}{c^2} - k^2\right)\right] E_z(x,y) = 0$$

$$E_z(x,y) = 0 \quad \text{at } x = 0, a \ , \ y = 0, b$$

To satisfy b.c. it is necessary that

$$\frac{\omega^2}{c^2} - k^2 = \gamma^2 > 0$$

for $\gamma^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$

$$E_z(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

TE modes

$$\left[\nabla_t^2 + \left(\frac{\omega^2}{c^2} - k^2\right)\right] B_z(x,y) = 0$$

$\frac{\partial B_z}{\partial x} = 0$ at $x = 0, a$, $\frac{\partial B_z}{\partial y} = 0$ at $y = 0, b$

for $\frac{\omega^2}{c^2} - k^2 = \gamma^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$

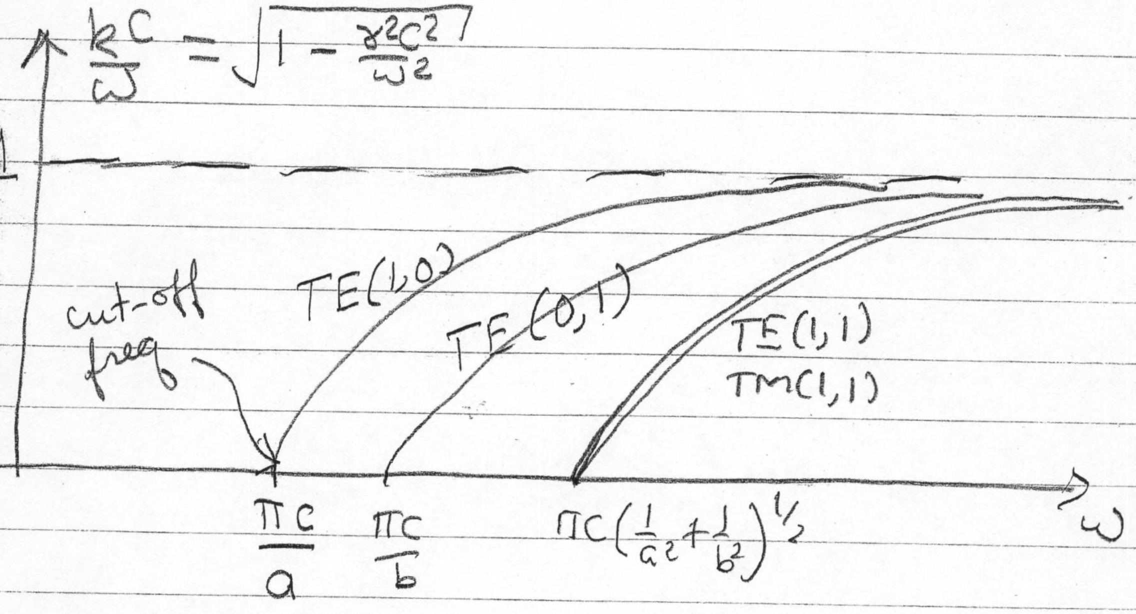
$$B_z(x,y) = \cos\frac{n\pi x}{a}\cos\frac{m\pi y}{b}$$

Cut-off frequency

for propagation it is necessary that

$$0 < k^2c^2 = \omega^2 - \gamma_{nm}^2 c^2$$

$$\therefore \quad \omega > \omega_c = \gamma_{min} c \qquad \text{cut-off freq.}$$



assume $a > b$

Homework

(24) Find TE and TM modes for a circular wave guide of radius R. Determine the cut-off frequency.

also D

group velocity + phase velocity

$$\frac{\omega^2}{c^2} = k^2 + \gamma^2$$

not limited to square cross section

$$\omega = \sqrt{\gamma^2 c^2 + k^2 c^2}$$

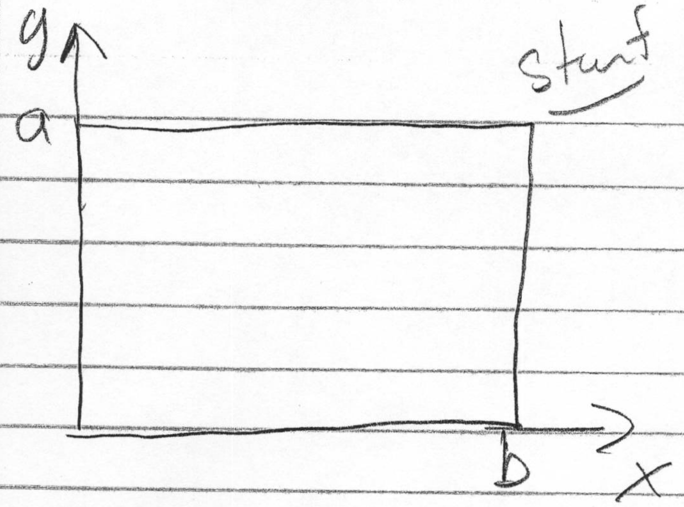
$$v_{ph} = \frac{\omega}{k} = \sqrt{c^2 + \frac{\gamma^2 c^2}{k^2}}$$

$$v_g = \frac{d\omega}{dk} = \frac{k c^2}{\sqrt{\gamma^2 c^2 + k^2 c^2}}$$

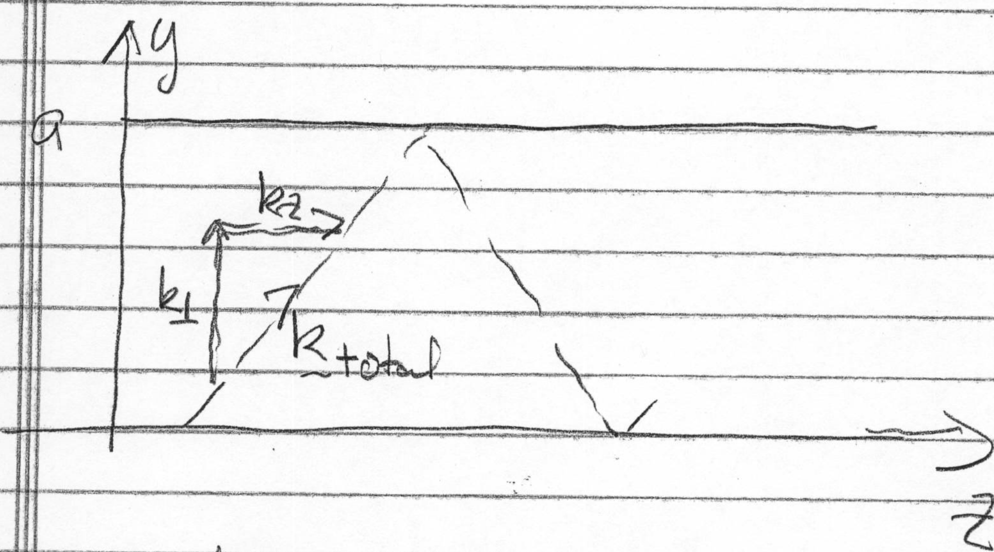
$$v_g v_{ph} = c^2$$

(for wave guides, not general result)

Start



let $b \to \infty$



$$k_\perp = \frac{\pi n}{a} \Leftarrow$$ standing wave in $\perp$ direction, $\sin\left(\frac{\pi n y}{a}\right) e^{-i\omega t} e^{ikz}$

$$v_g = c\frac{k_z}{k_{total}} = \frac{c\, k_z}{\sqrt{\underbrace{k_\perp^2}_{``\gamma_{nm}^2"} + k_z^2}}$$

$$v_{ph} = \frac{\omega}{k_z} = c\sqrt{k_\perp^2 + k_z^2}\,/\,k_z$$

Energy flow and attenuation

$$\langle \underset{\sim}{S} \rangle = \frac{1}{2} \mathrm{Re} \frac{c}{4\pi} \underset{\sim}{E} \times \underset{\sim}{B}^*$$

write supports eqs on board & save

$$\langle S_z \rangle = \frac{c}{8\pi} \mathrm{Re} \left[E_x B_y^* - E_y B_x^* \right]$$

for TM modes

$$\langle S_z \rangle = \frac{c}{8\pi} \frac{k_z \omega}{c \gamma^4} \left[\left(\frac{\partial E_z}{\partial x} \right)^2 + \left(\frac{\partial E_z}{\partial y} \right)^2 \right]$$

$$\langle \underline{P} \rangle = \iint \langle S_z \rangle dA = \frac{\omega k_z}{8\pi \gamma^4} \iint_A \nabla_t E_z \cdot \nabla_t E_z^* \, dA$$

$$\underset{\sim}{\nabla_t} \cdot \left(E_z^* \underset{\sim}{\nabla_t} E_z \right) = \underset{\sim}{\nabla_t} E_z^* \cdot \underset{\sim}{\nabla_t} E_z + E_z^* \nabla_t^2 E_z$$

$$\langle \underline{P} \rangle = \frac{\omega k_z}{8\pi \gamma^4} \left\{ \underbrace{\oint_C d\ell \, E_z^* \frac{\partial E_z}{\partial n}}_{= 0} - \iint_A E_z^* \underbrace{\nabla_t^2 E_z}_{-\gamma^2 E_z} \right\}$$

$$\langle \underline{P} \rangle = \frac{\omega k}{\gamma^2} \iint_A \frac{|E_z|^2}{8\pi} dA$$

Homework

(25) Show that for TE modes

$$\langle P \rangle = \frac{\omega k}{\gamma^2} \oiint_A \frac{|B_z|^2}{8\pi} dA$$

(26) For TM modes define the time-average energy/length

$$\langle U \rangle = \frac{1}{16\pi} \oiint_A \left[|E_z|^2 + |\underset{\sim}{E}_t|^2 + |\underset{\sim}{B}_t|^2 \right] dA$$

Show that

$$\langle U \rangle = \frac{\omega^2}{c^2 \gamma^2} \oiint \frac{|E_z|^2}{8\pi} dA$$

$$\frac{\langle P \rangle}{\langle U \rangle} = V_g$$