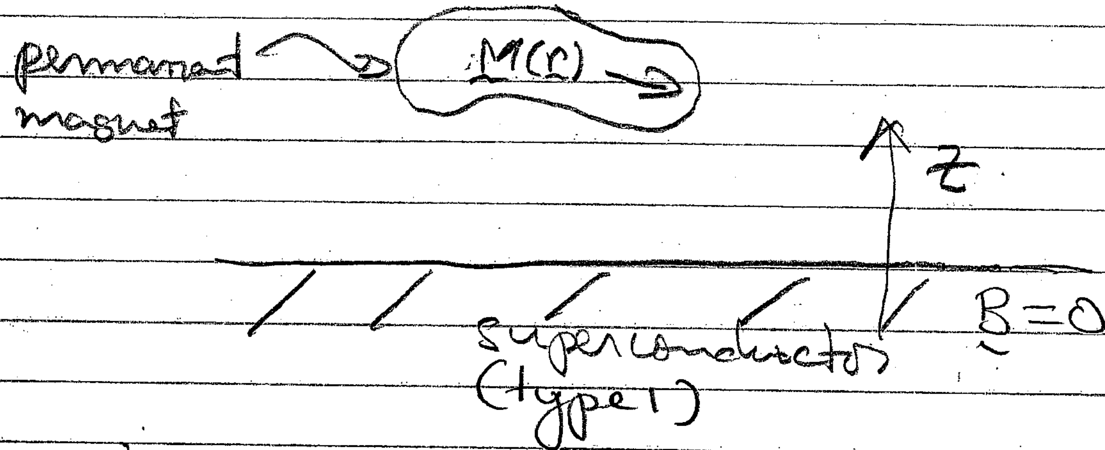
example (Neumann b.c.)



at interface

$$(B_n)_{vac} = (B_n)_{SC} = 0$$

$$(B_t)_{vac} \neq 0 \quad (\text{surface currents})$$

above superconductor

$$\nabla \times \underline{H} = \frac{4\pi}{c} \underline{J}_f = 0 \quad , \quad \underline{H} = -\nabla \varphi$$

$$\underline{B} = \underline{H} + 4\pi \underline{M}$$

$$0 = \nabla \cdot \underline{B} = -\nabla^2 \varphi + 4\pi \underbrace{\nabla \cdot \underline{M}}_{-\rho_B(\underline{r})}$$

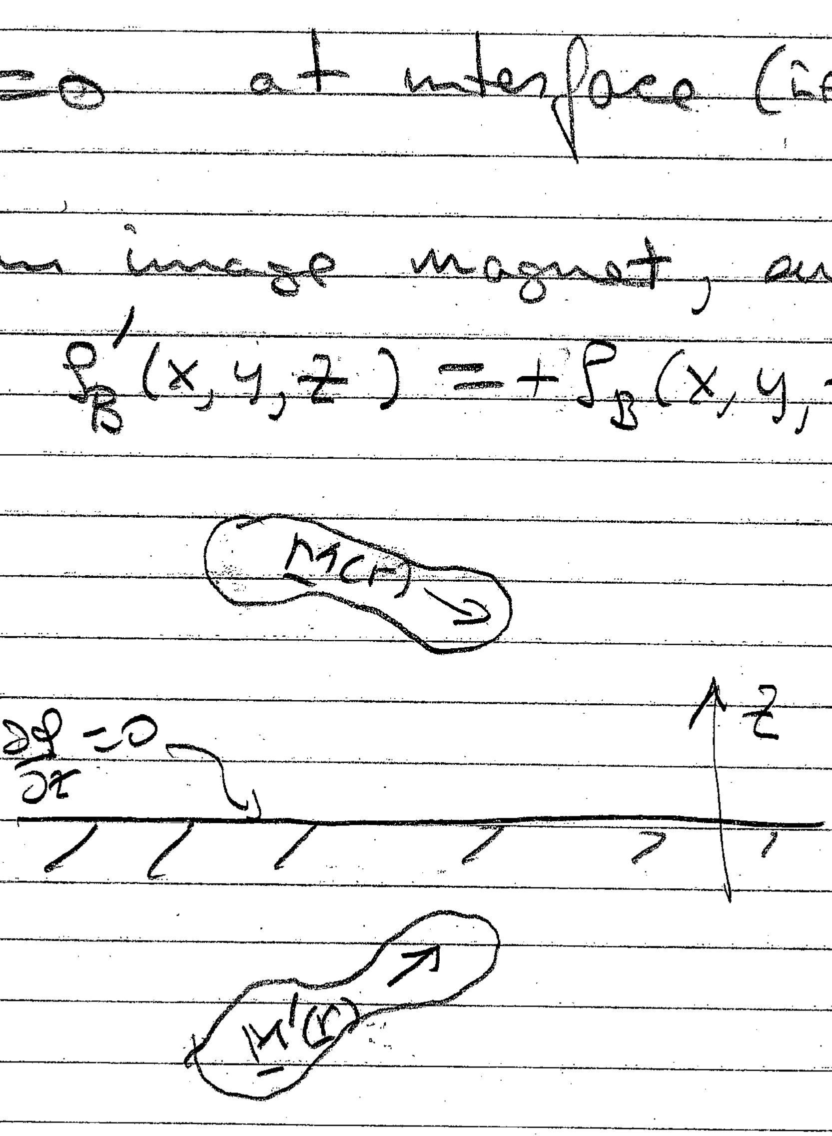
image potential

$$\varphi(\underline{r}) = \int d^3r' \frac{\rho_B(\underline{r}')}{|\underline{r}-\underline{r}'|} + \int d^3r' \frac{\rho'_B(\underline{r}')}{|\underline{r}-\underline{r}'|}$$

must choose image potential so that

$$B_n = -\frac{\partial\varphi}{\partial z} = 0 \quad \text{at interface (i.e., } z=0)$$

use an image magnet, such that $\rho'_B(x,y,z) = +\rho_B(x,y,-z)$

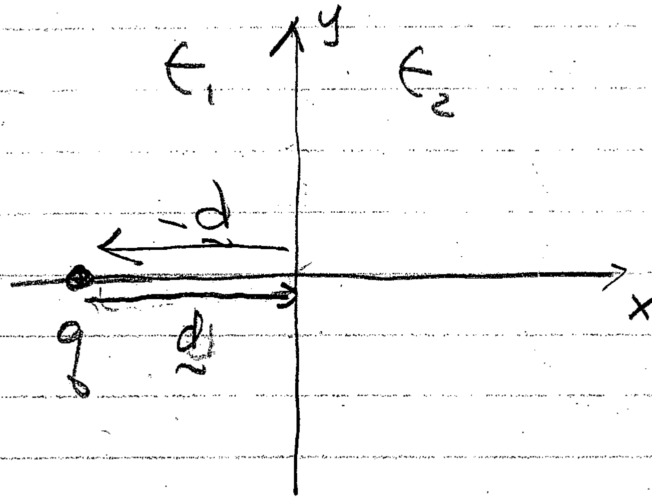


Note that the magnet is repelled by superconductor (floats above superconductor) (demonstrated by Ryan Baumbach)

$$\underset{\sim}{M}'(x,y,z) = [M_x(x,y,-z), M_y(x,y,-z), -M_z(x,y,-z)]$$

example

J. sec 4.4



$$\nabla \cdot \epsilon(\underset{\sim}{r}) \nabla \varphi = -4\pi q \, \delta(\underset{\sim}{r} + \underset{\sim}{d})$$

$$\varphi(\underset{\sim}{r}) \to 0 \quad \text{as } |\underset{\sim}{r}| \to \infty$$

let $\varphi(x,y,z) = \begin{cases} \varphi_1(x,y,z) & \text{for } x < 0 \\ \varphi_2(x,y,z) & \text{for } x > 0 \end{cases}$

$$\nabla^2 \varphi_1 = -\frac{4\pi q}{\epsilon_1} \delta(\underset{\sim}{r} + \underset{\sim}{d}), \quad \nabla^2 \varphi_2 = 0$$

$\nabla \cdot \epsilon \nabla \varphi$ at interface

at $x = 0$

$$\epsilon_1 \frac{\partial \varphi_1}{\partial x} = \epsilon_2 \frac{\partial \varphi_2}{\partial x} \qquad D_n \text{ cont.}$$

$$\varphi_1 = \varphi_2 \qquad E_t \text{ cont.}$$

image

let $$\varphi_1 = \frac{q}{\epsilon_1 |\underline{r}+\underline{d}|} + \frac{q'}{\epsilon_1 |\underline{r}-\underline{d}|}$$

image

$$\varphi_2 = \frac{q''}{\epsilon_2 |\underline{r}+\underline{d}|}$$

note that images are not in region where potential applies

$$\epsilon_1 \frac{\partial \varphi_1}{\partial x}\Big|_{x=0} = -\frac{q(\overset{=0}{x}+d)}{[y^2+z^2+(\underset{0}{\underset{\shortparallel}{x}}+d)^2]^{3/2}} - \frac{q'(\overset{=0}{x}-d)}{[y^2+z^2+(\underset{0}{\underset{\shortparallel}{x}}-d)^2]^{3/2}}$$

$$\epsilon_2 \frac{\partial \varphi_2}{\partial x}\Big|_{x=0} = \frac{-q''d}{[y^2+z^2+d^2]^{3/2}}$$

$$\epsilon_1 \frac{\partial \varphi_1}{\partial x} = \epsilon_2 \frac{\partial \varphi_2}{\partial x} \Rightarrow -q + q' = -q''$$

$$\varphi_1 = \varphi_2 \Rightarrow \frac{q}{\epsilon_1} + \frac{q'}{\epsilon_1} = \frac{q''}{\epsilon_2}$$

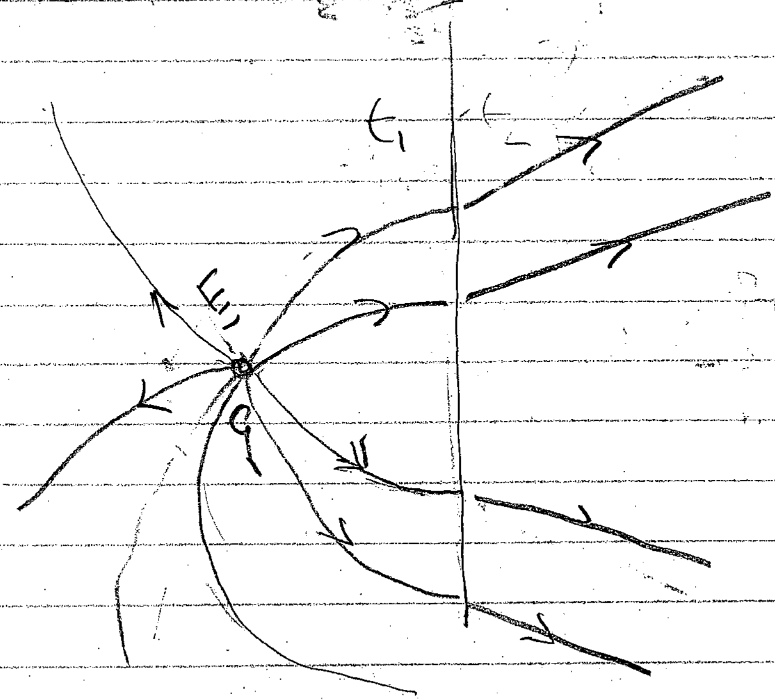
$$\therefore \quad \frac{q}{\epsilon_1} + \frac{q'}{\epsilon_1} = \frac{(q - q')}{\epsilon_2}$$

$$q\left[\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right] = -q'\left[\frac{1}{\epsilon_2} + \frac{1}{\epsilon_1}\right]$$

$$q' = -q\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\right)$$

$$q'' = q - q' = \frac{q\,2\epsilon_2}{\epsilon_1 + \epsilon_2}$$

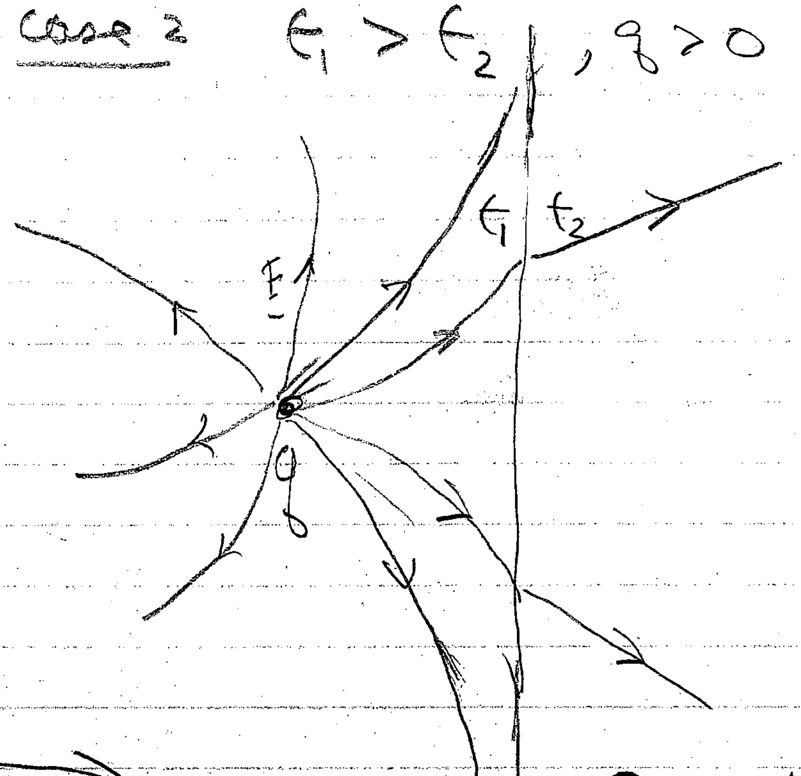
<u>case 1</u> $\epsilon_2 > \epsilon_1$, $q > 0$



q is attracted to dielectric 2

Case 2 $\epsilon_1 > \epsilon_2$, $q > 0$

q is repelled by dielectric 2



Study problems 1–5 in Sec 7 of L+L Vol. 8

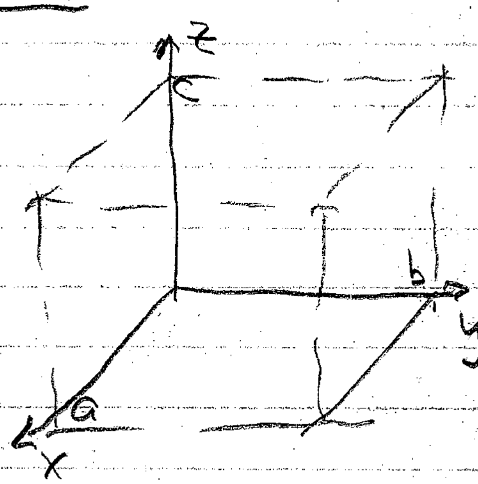
Counter example to Earnshaw's Theorem
(Chiao + Boyce, Phys. Rev. Lett. 25 3383 (1994))

Separation of variables (J. Chapters 2, 3)

rectangular coordinates

$\Phi = 0$ on all faces except top, where
$\Phi(x,y,c) = V(x,y)$

$\nabla^2 \Phi = 0$ inside



try sol'n

$$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$0 = \frac{1}{\Phi}\nabla^2\Phi = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-\beta^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{+\gamma^2}$$

$$-\alpha^2 - \beta^2 + \gamma^2 = 0$$

Because of b.c. on sides and bottom, let

$$X_n = \sin\left(\frac{n\pi}{a}x\right) \qquad \alpha_n^2 = \left(\frac{\pi n}{a}\right)^2$$

$$Y_n = \sin\left(\frac{m\pi y}{b}\right) \qquad \beta_m^2 = \left(\frac{\pi m}{a}\right)^2$$

$$Z_{n,m} = \sinh(\gamma_{nm} z) \quad , \quad \gamma_{nm}^2 = \frac{\pi^2 n^2}{a^2} + \frac{\pi^2 m^2}{b^2}$$

Because $\nabla^2\Phi = 0$ is linear, a superposition also is sol'n

$$\Phi(x,y,z) = \sum_{n,m} A_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh(\gamma_{nm} z)$$

b.c. on sides + bottom are satisfied

b.c. on top satisfied if

$$V(x,y) = \sum_{n,m} A_{n,m} \sinh(\gamma_{n,m} c) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

complete orthogonal set so
can satisfy b.c. on top

$$\int_0^a dx \int_0^b dy\, V(x,y) \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi y}{b}\right) =$$

$$A_{n'm'} \sinh(\gamma_{n'm'} c) \underbrace{\int_0^a dx \sin^2\left(\frac{\pi n' x}{a}\right)}_{a/2} \underbrace{\int_0^b dy \sin^2\left(\frac{\pi m' y}{b}\right)}_{b/2}$$

Suppose that

$$V(x,y) = V_0 \sin\left(\frac{n_0 \pi x}{a}\right) \sin\left(\frac{m_0 \pi y}{b}\right)$$

$$\Phi(x,y,z) = V_0 \sin\left(\frac{n_0 \pi x}{a}\right) \sin\left(\frac{m_0 \pi y}{b}\right) \frac{\sinh(\gamma_{n_0 m_0} z)}{\sinh(\gamma_{n_0 m_0} c)}$$

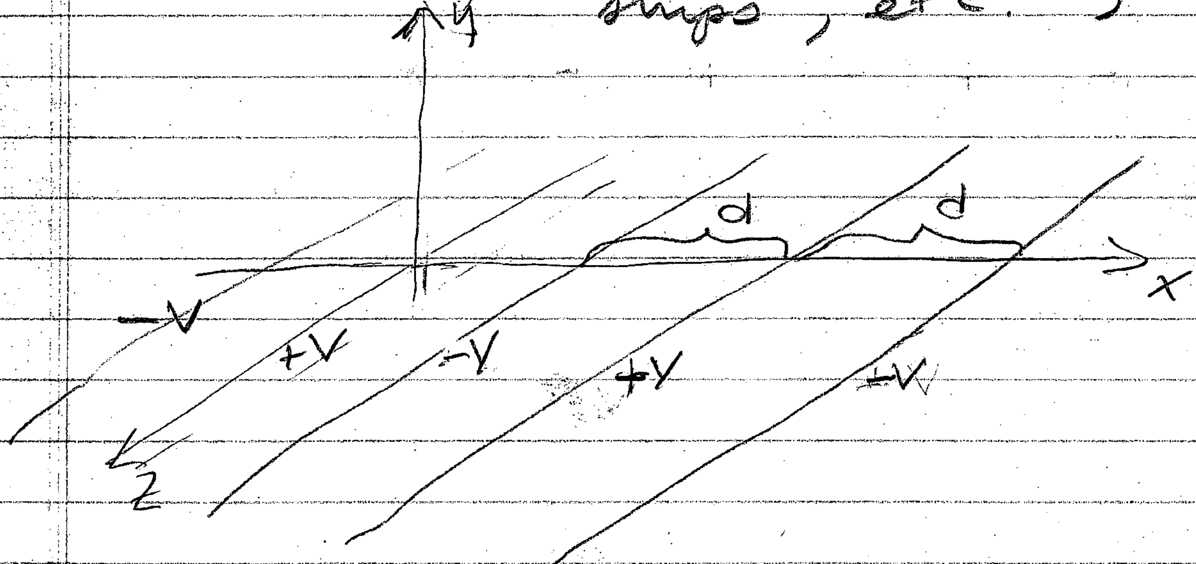
for $\gamma_{n_0 m_0} z >> 1$

$$\sqrt{\left(\frac{\pi n_0}{a}\right)^2 + \left(\frac{\pi m_0}{b}\right)^2}$$

$$\frac{\sinh(\gamma_{n_0 m_0} z)}{\sinh(\gamma_{n_0 m_0} c)} \simeq e^{-\gamma_{n_0 m_0}(c-z)}$$

exponential fall off of φ

example (alternately charged wires, strips, etc.)



drop $e^{\frac{\pi n y}{d}}$

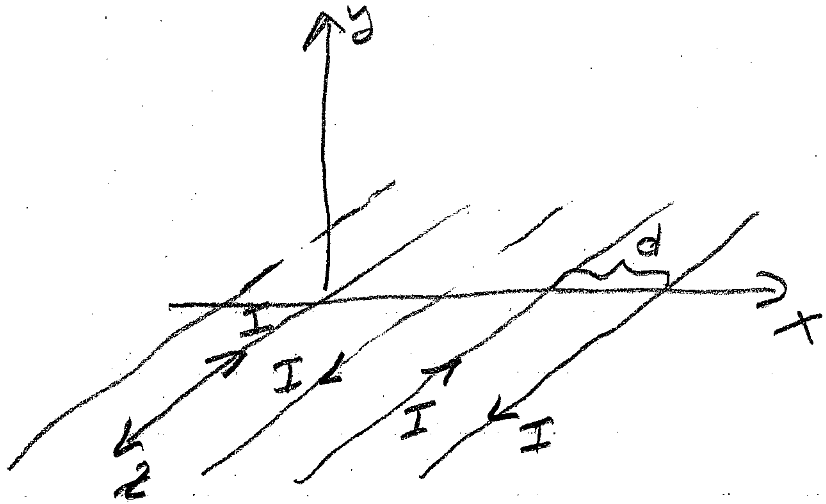
for $y > 0$

$$\varphi(x,y,z) = \sum_n A_n \cos\left(\frac{n\pi x}{d}\right)\left[e^{-\frac{\pi n y}{d}}\right]$$

$$\varphi(x,y,z) \sim \cos\frac{\pi}{d}x\, e^{-\frac{\pi y}{d}} \quad \text{for } y >> d$$

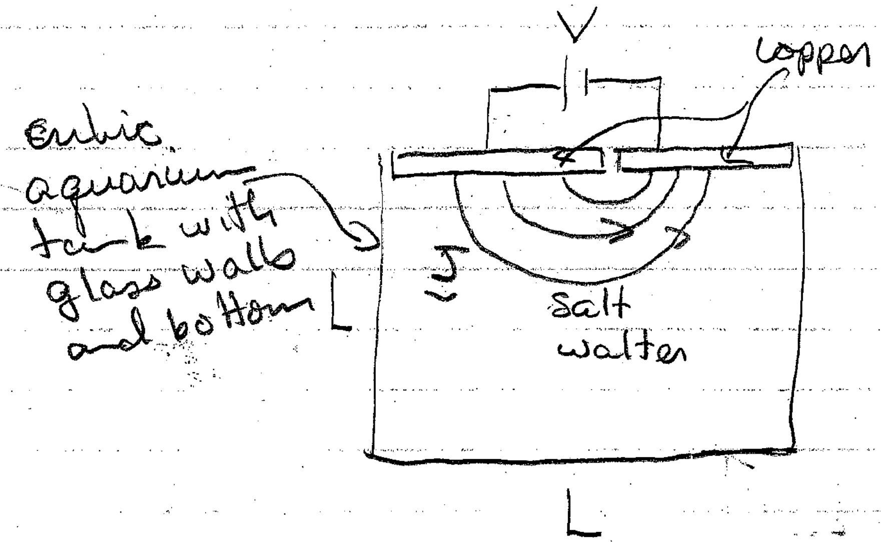
Homework Start ↑ review

(16)



Determine large y functional form of $\underline{B}(x, y)$

(17)

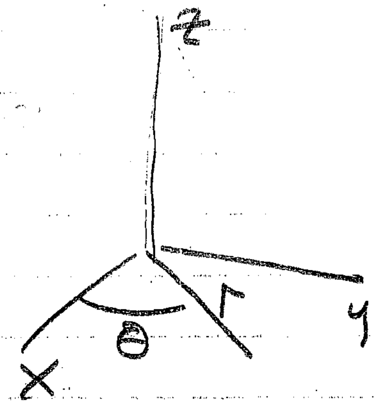


Determine current distribution in salt water

(Discuss b.c.)

Cylindrical coordinates (2D case)

$$\frac{\partial \varphi}{\partial z} = 0$$



$$\nabla^2 \varphi = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}$$

$$\varphi = R(r)\Theta(\theta)$$

$$0 = \frac{1}{\varphi}\nabla^2 \varphi = \frac{1}{rR}\frac{d}{dr} r \frac{dR}{dr} + \frac{1}{r^2}\underbrace{\left[\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}\right]}_{= -\alpha^2}$$

$$\Theta(\theta + 2\pi) = \Theta(\theta)$$

$$\Theta(\theta) = \cos l\theta, \ \sin l\theta \quad , \ \alpha = l$$

require periodic in θ

$$0 = \frac{1}{rR_l}\frac{d}{dr} r \frac{dR_l}{dr} - \frac{l^2}{r^2}$$

$$R_l = r^l, \ \frac{1}{r^{l+1}} \quad \text{for } l \neq 0$$

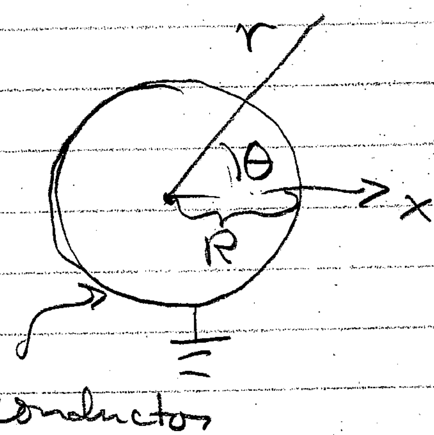
for $l = 0$

$$R_l = a\ ,\ b\ln\frac{r}{r_0}$$

$$\varphi(r,\theta) = a_0 + b_0 \ln r/r_0 + \sum_{l=1}^{\infty}\left(a_l r^l + b_l/r^l\right)\cos l\theta$$

$$+ \sum_{l=1}^{\infty}\left(c_l r^l + d_l/r^l\right)\sin l\theta$$

<u>example</u>



$\underset{\sim}{E} = E_0\hat{x}$, $\varphi = -E_0 x$

as $r \to \infty$

$\therefore\ \varphi(r,\theta) \simeq -E_0 \underbrace{r\cos\theta}_{x} + C$

as $r \to \infty$

require that $\nabla^2\varphi = 0$ for $r > R$

$$\varphi(R,\theta) = 0$$

look at general soln and eliminate terms

try

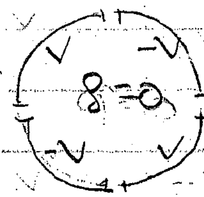
due to surface charge

$$\varphi(r,\theta) = a_1 r\cos\theta + \frac{b_1}{r}\cos\theta + a_0$$

$$a_1 = -E_0 \quad , \quad b_1 = E_0 R^2 \quad , \quad a_0 = 0$$

draw field lines

Homework



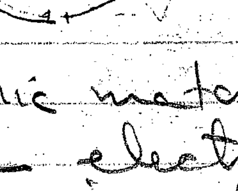
discuss terms $\ln\frac{r}{r_0}$ + $\frac{1}{r}\cos\theta$

(8) J, 2.14(a)

see J. page 149 + page 199 for same problem with spheres

(9) A cylinder of dielectric material ϵ and radius a is placed in uniform electric field $\underline{E} = E_0\hat{x}$. Determine $\underline{E}(\underline{r})$ in all of space. Sketch field.

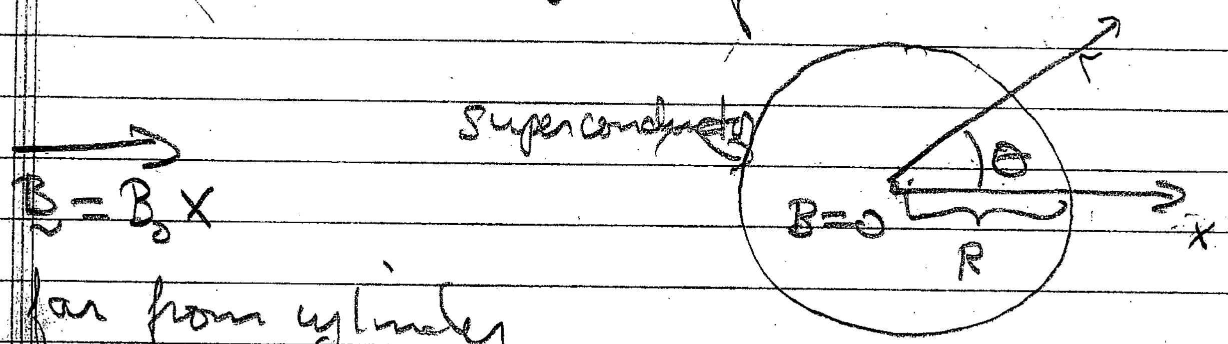
(10) A cylindrical shell of permeability $\mu >> 1$ is placed in a uniform external magnetic field $\underline{B} = B_0\hat{x}$. Determine $\underline{B}$ in all of space. Show that $\underline{B}$ is shielded out of interior was $\mu \to \infty$.



(11) read J sec 2.11

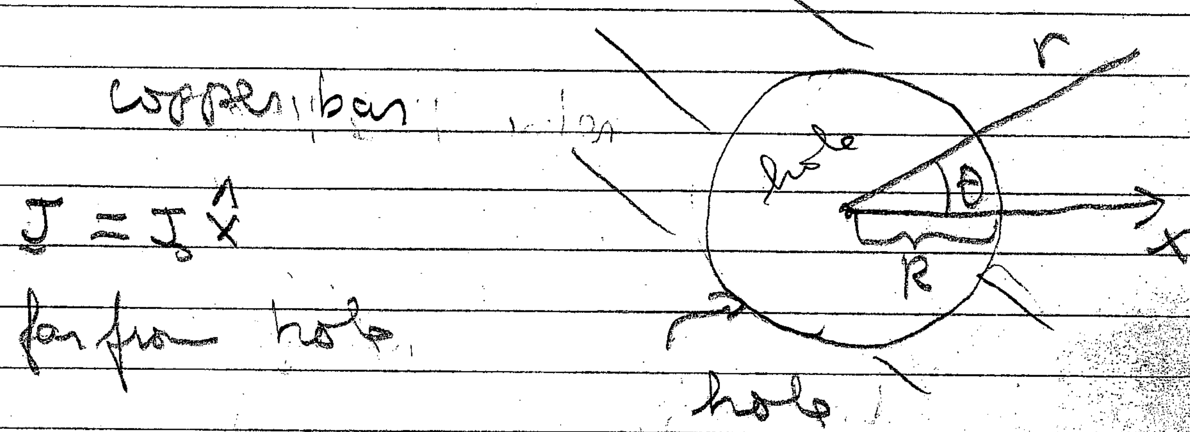
(11.)

(a) Calculate magnetic field near a superconducting cylinder that is arranged transverse to a uniform external magnetic field $\underline{B} = B_0 \hat{x}$.



(discuss b.c. on φ, where $\underline{B} = -\underline{\nabla}\varphi$)

(b) Calculate steady state current distribution near a cylindrical hole drilled in a current carrying bar of copper



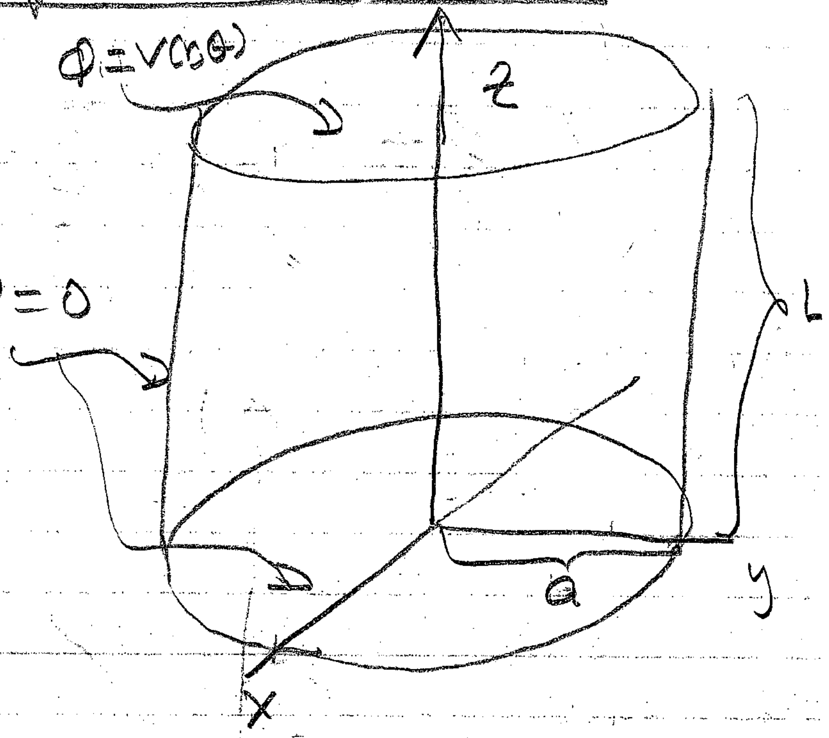
(discuss b.c. on φ where $\underset{\sim}{J} = -\sigma\nabla\varphi$)

4/17/03

 skip 37

Cylindrical problems (3D case)

J. sec 3.7



$\nabla^2 \Phi = 0$

$\Phi = R(r)\,\Theta(\theta)\,Z(z)$

$$0 = \frac{1}{\Phi}\nabla^2\Phi = \frac{1}{R}\frac{1}{r}\frac{d}{dr} r\frac{dR}{dr} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2}$$

$\Theta(\theta + 2\pi) = \Theta(\theta)$

try $\Theta(\theta) = \cos(l\theta)$, $\sin(l\theta)$

$Z(z=0)$

try $Z(z) = \sinh kz$

then

$$\frac{1}{r}\frac{d}{dr} r\frac{dR}{dr} + \left(k^2 - \frac{l^2}{r^2}\right)R = 0$$

Note that we could not have satisfied b.c. at $r=a$ [i.e. $\Phi(r=a)=0$] if we had taken $Z(z) = \sin kz$

$$R = J_l(kr)\,,\ N_l(kr)$$

Bessel functions of first and second kind

(first course) Start problem on wire & line charge

as $x \rightarrow 0$

$$J_l(x) \simeq \frac{1}{l!}\left(\frac{x}{2}\right)^l$$

$$N_l(x) \simeq \begin{cases} \frac{2}{\pi}\ln\frac{x}{2} & l=0 \\ -\frac{(l-1)!}{\pi}\left(\frac{2}{x}\right)^l & l \neq 0 \end{cases}$$

comment on relation to 2D

For this problem, exclude N_l

(Would require N_l for region between two concentric cylinders)

Choose k so that b.c. at $r=a$ is satisfied

$$R = J_l(ka) = 0$$

$ka = x_{ln}$ roots of Bessel function

$l=0 \quad x_{0n} \simeq 2.405, \ 5.520 \ - -$

$l=1 \quad x_{1n} \simeq 3.832, \ 7.016 \ - -$

see J. page 114

$$R_{ln}(r) = J_l\left(\frac{x_{ln}\, r}{a}\right)$$

complete orthogonal set on interval $(0, a)$

$$\Phi(r,\theta,z) = \sum_{l,n} J_l\left[\frac{x_{ln}}{a} r\right] \sinh\left(\frac{x_{ln}}{a} z\right) \cdot \left[A_{ln}\cos l\theta + B_{ln}\sin l\theta\right]$$

b.c. on sides and bottom are satisfied.

b.c. on top is satisfied if

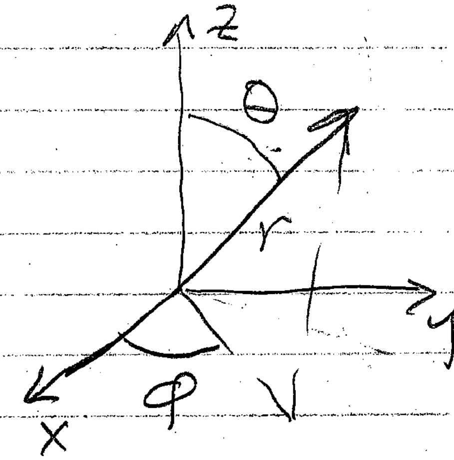
$$V(r,\theta) = \Phi(r,\theta,z=L)$$

$$\therefore \int_0^{2\pi} d\theta \cos(l\theta) \int_0^a r\,dr\, J_l\left(\frac{x_{ln}}{a} r\right) V(r,\theta)$$

$$= A_{ln} \underbrace{\int_0^{2\pi} d\theta \cos^2 l\theta}_{\pi} \underbrace{\int_0^a r\,dr\, J_l^2\left(\frac{x_{ln}}{a} r\right)}_{\frac{a^2}{2} J_{l+1}^2(x_{ln})} \sinh\left(\frac{x_{ln}L}{a}\right)$$

etc.

Spherical Geometry



$\Phi = \Phi(r, \theta, \varphi)$

potential

Spherical harmonics

$$Y_{l,m}(\theta, \varphi) \equiv \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}$$

associated Legendre function

defined for $l = 0, 1, 2, \ldots$

$m = -l, \ldots 0, \ldots l$

$$\underbrace{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]}_{-\frac{1}{\hbar^2}L^2} Y_{l,m} = -l(l+1)\, Y_{l,m}$$

$$\frac{1}{\hbar} L_z Y_{lm} = \frac{\partial}{i\partial\varphi} Y_{lm} = m Y_{lm}$$

The $Y_{lm}(\theta,\phi)$ are a complete ortho-normal set

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{l,m} Y_{l,m}(\theta,\phi)$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, Y_{l,m}(\theta,\phi) Y^*_{l',m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}$$

look for solns of

$\nabla^2 \Phi = 0$

let $\Phi(r,\theta,\phi) = \sum_{l,m} R_{l,m}(r) Y_{l,m}(\theta,\phi)$

$$0 = \nabla^2 \Phi = \sum_{l,m} \left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} \right.$$

$$\left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right] R_{lm}(r) Y_{lm}(\theta,\phi)$$

$$= \sum \left[\frac{1}{r^2}\frac{d}{dr} r^2 \frac{dR_{lm}}{dr} - \frac{l(l+1)R_{lm}}{r^2} \right] Y_{l,m}(\theta,\phi)$$

by orthogonality

$$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{dR_{\ell m}}{dr} - \frac{l(l+1)}{r^2} R_{\ell m} = 0$$

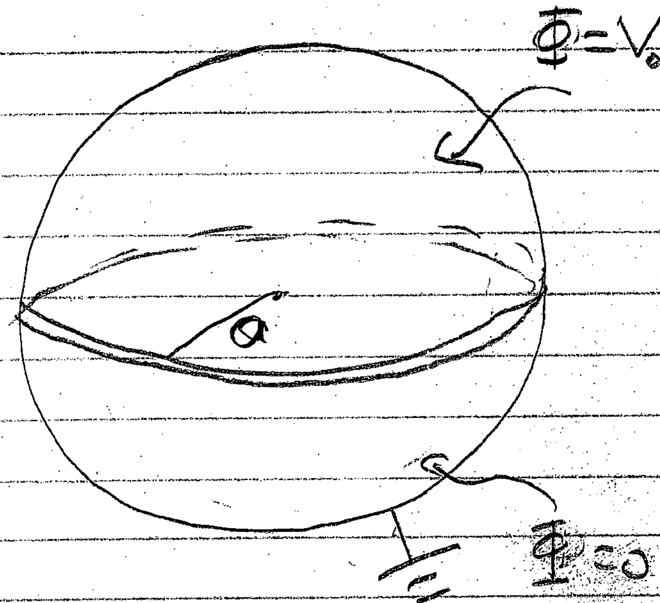
$$\therefore \quad R_{\ell m} = r^l, \ \frac{1}{r^{l+1}} \quad \text{for all } l$$

$$\Phi = \sum_{\ell m} \left[A_{\ell m} r^l + B_{\ell m}/r^{l+1} \right] Y_{l,m}(\theta, \varphi)$$

(i) comment on r^l dependence
(ii) ask class how to proceed to find Φ outside sphere

<u>example</u>

find Φ
inside sphere



$$\Phi = \sum_{l,m} A_{l,m} r^l Y_{l,m}(\theta, \varphi)$$

$$a^l A_{l,m} = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta \, Y_{l,m}(\theta,\phi) \, V_0$$

$$= \delta_{m,0} \, 2\pi V_0 \int_0^1 dx \underbrace{\sqrt{\frac{2l+1}{4\pi}} P_l(\overset{\cos\theta}{\overset{\parallel}{x}})}_{Y_{l,0}}$$

Rodrigue's formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l$$

See D. page 88

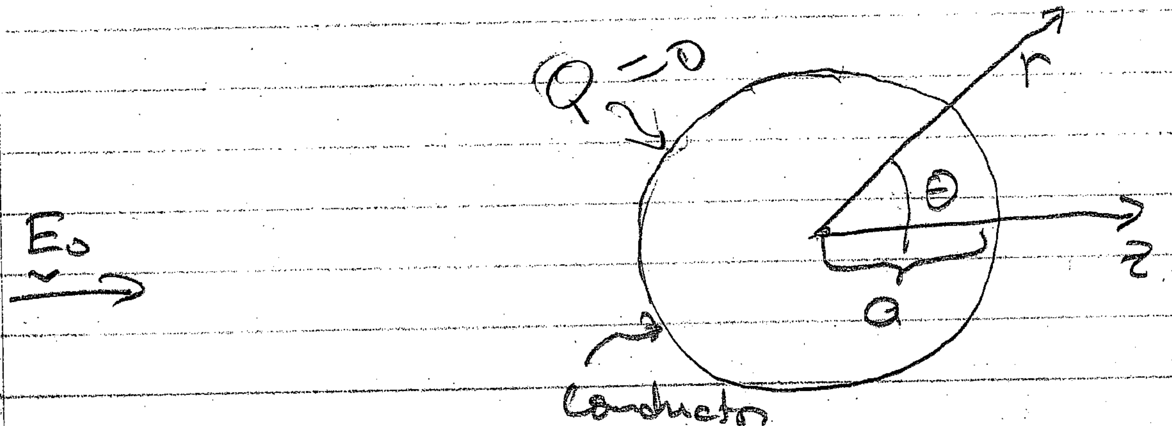
$$\int_0^1 dx \, P_l(x) = \frac{1}{2} \quad \text{for } l = 0$$

$$\int_0^1 dx \, P_l(x) = \frac{1}{2^l l!} \frac{d^{(l-1)}}{dx^{(l-1)}} (x^2-1)^l \Big|_0^1$$

$$= \begin{cases} 0 & \text{for even } l \\ \left(-\frac{1}{2}\right)^{(l-1)/2} \dfrac{(l-2)(l-4)(l-6)\cdots}{2\left(\frac{l+1}{2}\right)!} & \text{for odd } l \end{cases}$$

example

Unchanged conducting sphere is placed in uniform external field $\underset{\sim}{E_0}$



$$\frac{\partial \Phi}{\partial \varphi} = 0 \text{ by symmetry}$$

$$\Phi(r,\theta) = \sum_l \left(A_l r^l + B_l / r^{l+1}\right) P_l[\cos\theta]$$

as $r \to \infty$ $\quad \Phi \to -E_0 z = -E_0 r\cos\theta$

since $P_1(\cos\theta) = \cos\theta$, can set

$A_l = 0$ for $l \geq 2$

$A_1 = -E_0$

$$\Phi(r,\theta) = \left(A_0 + \frac{B_0}{r}\right) \underbrace{P_0(\cos\theta)}_{=1} + \left(-E_0 r + \frac{B_1}{r^2}\right) \underbrace{P_1(\cos\theta)}_{=\cos\theta}$$

$$+ \sum_{l \geq 2} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$\Phi(a,\theta) = \text{const.}$ implies that

$B_1 = E_0 a^3$, $B_l = 0$ for $l \geq 2$

have used orthogonality of P_l

$4\pi Q = -\iint d\underset{\sim}{s} \frac{\partial \Phi}{\partial r}$ implies that $B_0 = Q$

$$\therefore \Phi(r,\theta) = \overset{=0}{\frac{Q}{r}} + \left(-E_0 r + \frac{E_0 a^3}{r^2}\right) \cos\theta$$

$$+ A_0$$