Lecture 11

Partial Differential Equations in Physics

Fluid Mechanics
Electromagnetism
Quantum Mechanics

$$\frac{\partial}{\partial t}\left(\operatorname{curl}\vec{v}\right) = \operatorname{curl}\left(\vec{v} \times \operatorname{curl}\vec{v}\right)$$

$v_n = 0$ on solid surface boundary

Euler Equation for ideal fluid

non-linear time-dependent PDE

first order in time, second order in spatial derivatives

$\vec{v}(x, y, z, t)$ velocity field

$$\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} p\right) = -4\pi G \rho$$

hydrostatics

Navier-Stokes viscosity

$$\Delta \phi = - \rho$$

Poisson Eq. of electrostatics

elliptic PDE

boundary conditions on ϕ

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{\partial^2 \vec{E}}{\partial x^2} = 0$$

EM wave

hyperbolic PDE

$$\Delta \vec{E} = 0$$

eigenvalue problem in cavity

$$\Delta \vec{B}$$

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2 \psi + V \cdot \psi$$

parabolic PDE

$$- \frac{\hbar^2}{2m} \Delta \psi_n + V \psi_n = E_n \psi_n$$

QM eigenvalue problem

Elliptic PDE

Ideal Fluid PDE

fluid – liquid or gas (difference in density)

continuous medium : any small volume element has large number of molecules

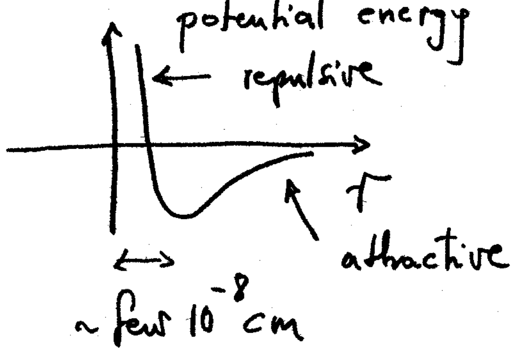
"infinitesimal" volume element large compared with intermolecular distance

smallest instrument probes

10^{-3} cm

$\sim 10^{10}$ particles

potential energy between molecules



fluid particle (point in fluid) painted

$\vec{V}\,(x, y, z, t)$ velocity field

$p\,(x, y, z, t)$ pressure

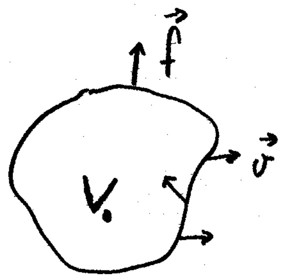
$\rho\,(x, y, z, t)$ density

determines thermodynamic state of fluid

Fundamental equations:

$$\oint \rho \cdot \vec{v} \cdot d\vec{f} \qquad \int \rho \, dV$$

total mass in volume V_0



total mass of fluid flowing out of volume V_0

$$\frac{\partial}{\partial t} \int \rho \, dV = - \oint \rho \vec{v} \, d\vec{f}$$

$$\downarrow$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \cdot \vec{v}) = 0$$ continuity eq.

$$\downarrow$$

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} \vec{v} + \vec{v} \cdot \operatorname{grad} \rho = 0$$

$\vec{j} = \rho \cdot \vec{v}$ mass flux density

Euler's Equation

$- \oint p \, d\vec{f}$ pressure over the surface boundary of the volume

$$\hookrightarrow - \int \operatorname{grad} p \, dV$$

$-\,\mathrm{grad}\,p \cdot dV$ force on volume element

$$\rho \cdot \frac{d\vec{v}}{dt} = -\,\mathrm{grad}\,p$$

$\frac{d\vec{v}}{dt}$ rate of change in velocity of fluid particle

Not $\frac{\partial \vec{v}}{\partial t}$ at fixed point in space!

two types of forces in fluids:

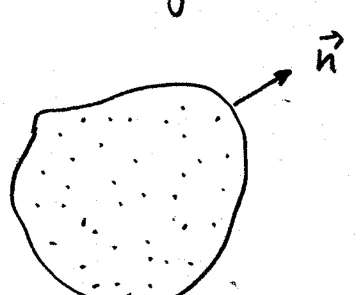
long range – gravity
electromagnetic
centrifugal (rotating fluid)

Uniform on infinitesimal volume element

$\vec{F}(\vec{x}, t) \cdot \rho \, dV$

$\vec{F} = \vec{g}$ gravity on earth

short range molecular origin



$p \cdot \vec{n}$ force per unit area
same in all directions

$$d\vec{v} = \left(\frac{\partial \vec{v}}{\partial t}\right) dt + (d\vec{r} \cdot \mathrm{grad})\,\vec{v}$$

$$dx \frac{\partial \vec{v}}{\partial x} + dy \frac{\partial \vec{v}}{\partial y} + dz \frac{\partial \vec{v}}{\partial z} = (d\vec{r} \cdot \mathrm{grad})\,\vec{v}$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \mathrm{grad})\,\vec{v}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \mathrm{grad})\,\vec{v} = -\frac{1}{\rho}\,\mathrm{grad}\,p$$

Euler eq.

In gravitational field $\rho \vec{g}$ is additional gravitational force:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \mathrm{grad})\,\vec{v} = -\frac{\mathrm{grad}\,p}{\rho} + \vec{g}$$

ideal fluid — no friction, viscosity negligible

thermal conductivity and viscosity are unimportant — no heat exchange

Start from Euler's eq. and write it in a form which involves velocity vector only:

Use the identity:

$$\frac{1}{2}\,\text{grad}\, v^2 = \vec{v} \times \text{curl}\,\vec{v} + (\vec{v}\cdot\text{grad})\,\vec{v}$$

If there is a one-to-one relation between p and ρ ($S(p,\rho)$ = constant for isentropic flow) then $-\frac{1}{\rho}\,\text{grad}\,\vec{p}$ can be written as the gradient of some function:

$$\frac{1}{\rho}\,\text{grad}\, p = \text{grad}\, W$$

W enthalpy (heat function per unit mass)

$$dW = \frac{dp}{\rho}$$

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times \text{curl}\,\vec{v} = -\,\text{grad}\left(W + \frac{1}{2}v^2\right)$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\,\text{grad})\,\vec{v} = -\,\text{grad}\, W \qquad \frac{1}{2}\,\text{grad}(v^2) = \vec{v} \times \text{curl}\,\vec{v} + (\vec{v}\cdot\text{grad})\,\vec{v}$$

By taking the curl of both sides:

$$\frac{\partial}{\partial t}\left(\operatorname{curl}\vec{v}\right) = \operatorname{curl}\left(\vec{v}\times\operatorname{curl}\vec{v}\right)$$

Euler Eq. in terms of velocity vector field

Has to be supplemented by boundary conditions

$v_n = 0$ on solid surface boundary at rest

Hydrostatics

$\operatorname{grad} p = \rho\cdot\vec{g}$ fluid at rest in gravitational field

$\operatorname{grad} p = 0 \rightarrow p = \text{const}$ if no external force

Star or Galaxy: ϕ Newtonian gravitational potential

$$\Delta\phi = 4\pi G\rho$$

↑ Newtonian constant

$-\operatorname{grad}\phi$ gravitational acceleration

$-\rho\operatorname{grad}\phi$ gravitational force

$$\text{grad}\, p = -\rho\, \text{grad}\, \phi \quad \text{condition of equilibrium}$$

div $\frac{1}{\rho}\times$

$$\text{div}\left(\frac{1}{\rho}\,\text{grad}\, p\right) = -4\pi G\rho$$

hydrostatic equation

Bernoulli's equation

for steady flow $\frac{\partial \vec{v}}{\partial t} = 0$

$$\frac{1}{2}\,\text{grad}\, v^2 - \vec{v}\times \text{curl}\,\vec{v} = -\text{grad}\, W$$

streamlines : these are lines such that the tangent to a streamline at any point gives the direction of the velocity at that point

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z} \quad \text{equation for streamline}$$

In steady flow the streamlines do not vary in time, and coincide with the paths of the fluid particles. In non-steady flow this coincidence no longer occurs : the tangent to the streamlines gives the directions of the velocities of fluid particles at various points in space at a given instant, whereas the tangents to the paths give the directions of the velocities of given fluid particles at different times

$\vec{l}$ unit vector tangent to the streamlines at each point

$$\vec{l} \cdot \text{grad}\, W = \frac{\partial W}{\partial l}$$

$\vec{l} \cdot$ for steady state equation

$$\vec{l} \cdot (\vec{v} \times \text{curl}\, \vec{v}) = 0$$

$$\frac{\partial}{\partial l}\left(\frac{1}{2}v^2 + W\right) = 0$$

$\frac{1}{2}v^2 + W = \text{constant}$ along streamline

Bernoulli's equation

In gravity (along z direction)

$$\vec{l} \cdot \vec{g} = -g\frac{dz}{dl}$$

$$\frac{\partial}{\partial l}\left(\frac{1}{2}v^2 + W + gz\right) = 0$$

$$\frac{1}{2}v^2 + W + gz = \text{constant}$$