Physics 140A — Midterm 3 Solutions — Fall 2010

Problem 1

At $T=\infty$ all states are equally likely:

$$\bar{E} = \frac{0 + 3\times 2\varepsilon + 1\times 5\varepsilon}{1+3+1} = \frac{11\varepsilon}{5} = \boxed{2.2\,\varepsilon}$$

(b)

$$2\varepsilon = \frac{3\times 2\varepsilon \times e^{-2\beta\varepsilon} + 1\times 5\varepsilon\times e^{-5\beta\varepsilon}}{1 + 3\times e^{-2\beta\varepsilon} + 1\times e^{-5\beta\varepsilon}} \Rightarrow$$

$$\Rightarrow 2\varepsilon + \cancel{6\varepsilon e^{-2\beta\varepsilon}} + 2\varepsilon e^{-5\beta\varepsilon} = \cancel{6\varepsilon e^{-2\beta\varepsilon}} + 5\varepsilon e^{-5\beta\varepsilon} \Rightarrow$$

$$\Rightarrow 3\varepsilon e^{-5\beta\varepsilon} = 2\varepsilon \Rightarrow e^{-5\beta\varepsilon} = \frac{2}{3} \Rightarrow \frac{5\varepsilon}{kT} = \ln\frac{3}{2} \Rightarrow kT = \frac{5\varepsilon}{\ln 3/2}$$

$$\Rightarrow \boxed{kT = 2.466\,\varepsilon \cdot 5 = 12.33\,\varepsilon}$$

c) $P(0) = \frac{1}{Z}$, $P(2\varepsilon) = \frac{3e^{-2\beta\varepsilon}}{Z} \Rightarrow e^{-2\beta\varepsilon} = \frac{1}{3} \Rightarrow$

$$\Rightarrow \frac{2\varepsilon}{kT} = \ln 3 \Rightarrow kT = \frac{2}{\ln 3}\varepsilon \Rightarrow \boxed{kT = 1.82\,\varepsilon}$$

(d) $F = -kT\ln Z$; at $kT = 5\varepsilon$:

$$Z = 1 + 3e^{-2\beta\varepsilon} + e^{-5\beta\varepsilon} = 1 + 3e^{-\frac{2}{5}} + e^{-1} = 3.3788$$

$$F = -5\varepsilon\times 1.2175 \Rightarrow \boxed{F = -6.088\,\varepsilon}$$

Problem 2

(a) Since $kT = 10\,\varepsilon_r >> \varepsilon_r$, the classical result should be valid $\Rightarrow$

$$\boxed{\bar{E}_{rot} = kT = 0.02\,eV}$$

For vibrations, the average energy is

$$\bar{E}_{vib} = \frac{\varepsilon_v}{e^{\beta\varepsilon_v}-1} \quad \text{with } \varepsilon_v = 0.04\,eV \Rightarrow \beta\varepsilon_v = 2 \Rightarrow$$

$$\boxed{\bar{E}_{vib} = \frac{0.04\,eV}{e^2-1} = 0.0063\,eV}$$

(b) Rotational partition function is

$$Z_{rot} = \sum_{j=0}^{\infty} (2j+1)\, e^{-j(j+1)\varepsilon_r/kT}$$

For $kT = 0.002\,eV \Rightarrow kT = \varepsilon_r$. So $\varepsilon_r/kT = 1 \Rightarrow$

$$Z_{rot} = 1 \cdot e^{-0} + 3e^{-2} + 5e^{-6} + 7e^{-12} + \cdots$$

$$= 1 + 0.4060 + 0.01239 + 0.00004 + \cdots$$

$$\Rightarrow \boxed{Z = 1.418}$$

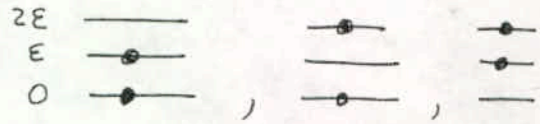
(c) $$C_v(vib) = \frac{\partial \bar{E}_{vib}}{\partial T} = \frac{1}{kT^2}\frac{\varepsilon_v^2}{(e^{\beta\varepsilon_v}-1)^2} = \left(\frac{\varepsilon_v}{kT}\right)^2 \frac{e^{\beta\varepsilon_v}}{(e^{\beta\varepsilon_v}-1)^2}\cdot k$$

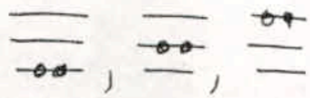
$\Rightarrow$ (i) $kT = 0.08\,eV = 2\,\varepsilon_v \Rightarrow C_v = \dfrac{1.65 \cdot 0.5^2}{(e^{0.5}-1)^2}\,k = \boxed{0.979\,k}$

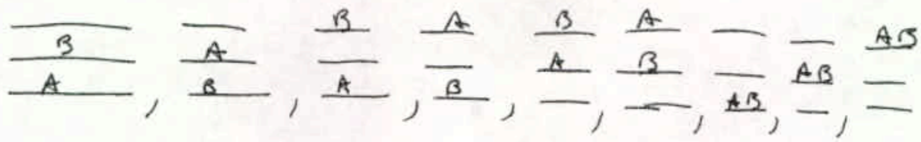
(ii) $kT = 0.8\,eV = 20\,\varepsilon_v \Rightarrow C_v = \dfrac{(1/20)^2 \cdot 1.05\,k}{(e^{1/20}-1)^2} = \boxed{0.9998\,k}$

(iii) $kT >>> \varepsilon_v \Rightarrow C_v = \left(\dfrac{\varepsilon_v}{kT}\right)^2 \dfrac{e^{\varepsilon_v/kT}}{(\varepsilon_v/kT)^2}\,k = \boxed{k}$ (using $e^x \approx x+1$ for $x<<1$)

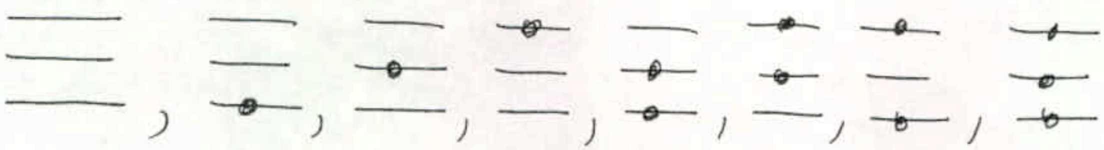
Problem 3

Fermions:  $\boxed{= 3 \text{ states}}$

Bosons: same as above, plus  $\boxed{= 6 \text{ states}}$

Classical:  $= \boxed{9 \text{ states}}$

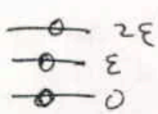
(b) Fermions: possible states are

 $= \boxed{8 \text{ states}}$

(c) $E_{tot} = 0$ if no particles in the system

1 particle, $T=0$:  $E_{tot} = 0$

2 particles, $T=0$:  $E_{tot} = \varepsilon$

3 particles, $T=0$:  $E_{tot} = 3\varepsilon$

(d) If there are no particles $\Rightarrow \mu < 0$

If there is 1 particle, $0 < \mu < \varepsilon$ $\Bigg\} \Rightarrow$ for $\boxed{\mu < \varepsilon, \; E_{tot} = 0}$

If there are 2 particles, $\boxed{\varepsilon < \mu < 2\varepsilon \qquad E_{tot} = \varepsilon}$

If there are 3 particles, $\boxed{\mu > 2\varepsilon \qquad E_{tot} = 3\varepsilon}$

based on the fact that $\dfrac{1}{e^{\beta(E-\mu)}+1}$ is $\left\{\begin{array}{l} 0 \text{ for } \mu < E \\ 1 \text{ for } \mu > E \end{array}\right\}$ at $T=0$

and E values are 0, ε and 2ε in this problem.