Physics 140A — Midterm 2 Solutions — Fall 2010

(a) $\underbrace{P_1 V_1 = N k T_1}_{\text{point 1}}$; $\underbrace{2P_1 \cdot V_1 = N k T_2}_{\text{point 2}}$; $\underbrace{P_1 \cdot 2V_1 = N k T_3}_{\text{point 3}} \Rightarrow \boxed{T_2 = T_3 = 2T_1}$

(b) Work = area enclosed = $\frac{\text{base} \times \text{height}}{2} \Rightarrow W = (2P_1 - P_1)(2V_1 - V_1)/2$

$\Rightarrow \boxed{W = P_1 V_1 / 2 = N k T_1 / 2}$

(c) $W_{12} = 0 \Rightarrow Q_{12} = \Delta U = U_2 - U_1 = \frac{3}{2} N k (T_2 - T_1) \Rightarrow \boxed{Q_{12} = \frac{3}{2} N k T_1}$

(d) $T_2 = T_3 \Rightarrow \Delta U = 0 \Rightarrow Q_{23} = W_{23}$ = area of trapezoid =

$= \frac{2P_1 + P_1}{2} \cdot (2V_1 - V_1) = \frac{3}{2} P_1 V_1 = \boxed{\frac{3}{2} N k T_1 = Q_{23}}$

(e) $Q_{abs} = Q_{12} + Q_{23} = 3 N k T_1$, $\boxed{e = \frac{W}{Q_{abs}} = \frac{1}{6} = 16.66\%}$

(f) The lowest temperature is $T_1 = T_c$

The temperature at points 2 and 3 = $2T_1$. If the Carnot engine operates between T_1 and $2T_1$, its efficiency is $\boxed{e = 1 - 1/2 = 50\%}$

However, the highest temperature in this cycle is <u>not</u> $2T_1$.

Find it from $\frac{2P_1 - P_1}{2V_1 - V_1} = \frac{P - P_1}{2V_1 - V}$, equation for the 2-3 line, gives for temperature along that line $T(V) = \frac{3T_1}{V_1} \cdot V - \frac{T_1}{V_1} V^2$, maximum is given by $\frac{\partial T}{\partial V} = 0 \Rightarrow V = \frac{3}{2} V_1 \Rightarrow P = \frac{3}{2} P_1 \Rightarrow \boxed{T = \frac{9}{4} T_1 = T_h}$

$\Rightarrow \boxed{e_{carnot} = 1 - \frac{T_c}{T_h} = 1 - \frac{4}{9} = \frac{5}{9} = 55.6\%}$

Problem 2

(a) $S(U,V) = Nk \ln \frac{U}{N} + Nk \ln \frac{V}{N}$ solve for U:

$$Nk \ln \frac{U}{N} = S - Nk \ln \frac{V}{N} \Rightarrow \ln \frac{U}{N} = \frac{S}{Nk} - \ln \frac{V}{N} \Rightarrow$$

$$\frac{U}{N} = \frac{N}{V} e^{S/Nk} \Rightarrow \boxed{U = \frac{N^2}{V} e^{S/Nk}}$$

(b) $dU = TdS - PdV \Rightarrow T = \left.\frac{\partial U}{\partial S}\right)_V , \quad P = -\left.\frac{\partial U}{\partial V}\right)_S$

$$T = \left.\frac{\partial U}{\partial S}\right)_V = \frac{N^2}{NkV} e^{S/Nk} \Rightarrow \boxed{T = \frac{N}{kV} e^{S/Nk}}$$

$$P = -\left.\frac{\partial U}{\partial V}\right)_S = -\left(-\frac{N^2}{V^2}\right) e^{S/Nk} \Rightarrow \boxed{P = \frac{N^2}{V^2} e^{S/Nk}}$$

(c) $\left.\frac{\partial T}{\partial V}\right)_S = -\frac{N}{kV^2} e^{S/Nk}$; $\left.\frac{\partial P}{\partial S}\right)_V = \frac{N^2}{NkV^2} e^{S/Nk} = \frac{N}{kV^2} e^{S/Nk}$

$$\Rightarrow \boxed{\left.\frac{\partial T}{\partial V}\right)_S = -\left.\frac{\partial P}{\partial S}\right)_V \quad \text{a Maxwell relation}}$$

(d) From T expression $\Rightarrow$ $e^{S/Nk} = \frac{kVT}{N}$, replacing in P expression,

$$P = \frac{N^2}{V^2} \frac{kVT}{N} \Rightarrow \boxed{P = \frac{NkT}{V}}$$

(e) $H = U + PV = \frac{N^2}{V} e^{S/Nk} + \frac{N^2}{V^2} e^{S/Nk} \cdot V \Rightarrow \boxed{H = \frac{2N^2}{V} e^{S/Nk}}$

(f) Comparing with $U = \frac{N^2}{V} e^{S/Nk} \Rightarrow H = 2U$, and $H = U + PV \Rightarrow$

$$\Rightarrow \boxed{U = PV}$$

Problem 3

$$P = \frac{NaT}{V} - a\frac{N^2}{V^2}, \quad U = 2NaT - a\frac{N^2}{V}$$

(a) $$dS = \frac{1}{T}dU + \frac{P}{T}dV$$

$$dU = 2Na\,dT + a\frac{N^2}{V^2}dV \Rightarrow$$

$$dS = \frac{1}{T}\left(2Na\,dT + a\frac{N^2}{V^2}dV\right) + \frac{1}{T}\left(\frac{NaT}{V} - a\frac{N^2}{V^2}\right)dV \Rightarrow$$

$$dS = \frac{2Na}{T}dT + \frac{aN^2}{TV^2}dV + \frac{Na}{V}dV - \frac{aN^2}{TV^2}dV \Rightarrow$$

$$\boxed{dS = \frac{2Na}{T}dT + \frac{Na}{V}dV}$$

(b) For a process with constant $S \Rightarrow dS = 0 \Rightarrow$

$$0 = \frac{2Na}{T}dT + \frac{Na}{V}dV \Rightarrow \frac{2}{T}dT + \frac{dV}{V} = 0 \Rightarrow$$

$$\Rightarrow 2\ln T + \ln V = \text{const} \Rightarrow \boxed{T^2 V = \text{const.}}$$

(c) Initially T_i, V_i; with $T_f = 4T_i \Rightarrow T_f^2 = 16T_i^2$;

$$T_i^2 V_i = T_f^2 V_f \Rightarrow T_i^2 V_i = 16\,T_i^2 V_f \Rightarrow$$

$$\Rightarrow \boxed{V_f = \frac{V_i}{16}}$$