Physics 110A: Midterm Exam #2

`Instructions: Do either problem. Do only one problem.`

[1] A point particle of mass m slides frictionlessly under the influence of gravity along a parabola. The parabola is defined by the equation $y = x^2/2b$, where b is a constant.

(a) Treat the confinement of the particle to the parabola as a constraint. Using generalized coordinates x and y, find the Lagrangian L.

(b) Find the equations of motion.

(c) Identify all conserved quantities.

(d) Assuming that the particle is released from rest at a location $x = x_0$, derive expressions for the forces of constraint Q_x and Q_y as a function of the horizontal coordinate x alone.

(e) Suppose that the mass m serves as the point of support of a pendulum bob of mass μ, suspended via a massless rigid rod of length ℓ. The pendulum is free to swing in the (x, y) plane. *Implementing the constraint $y = x^2/2b$ at the outset*, derive the Lagrangian $L(x, \theta, \dot{x}, \dot{\theta})$.

(f) Find expressions for the momenta p_x and p_θ.

[2] A pendulum consisting of a massless rigid rod of length ℓ and a point mass m is suspended from a hoop of radius a which lies in a horizontal plane. The point of suspension is driven to move around the hoop with constant angular velocity Ω. The pendulum's instantaneous motion is constrained to lie in a vertical plane bisecting the hoop. *Do not* use the method of undetermined multipliers. (Since the constraint is "holonomic" it may be implemented at the outset.)

(a) Using the single generalized coordinate θ, the instantaneous displacement of the pendulum rod from the vertical, write down the Lagrangian for the system.

(b) What is the Hamiltonian? Is H conserved? Explain why or why not.

(c) Write down the equations of motion.

(d) Show that the equations of motion may be written in the form $\ddot{\theta} = -\partial W_{\text{eff}}/\partial\theta$. Find the (scaled) effective potential, $W_{\text{eff}}(\theta)$.

(e) Derive an equation for the equilibrium value of θ_0, *i.e.* the value of θ for which the generalized force F_θ vanishes.

(f) Suppose $a = \ell/\sqrt{2}$ and furthermore that $\theta_0 = \frac{1}{4}\pi$. Solve then for Ω and for the frequency of small oscillations about equilibrium.

SOLUTIONS

[1] (a) $T = \frac{1}{2} m(\dot{x}^2+\dot{y}^2)$, $U = mgy$ $\Big\}$ $L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - mgy$

(b) Constraint:

$$G(x,y) = y - \frac{x^2}{2b} = 0$$

Thus,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial q_\sigma} \qquad (k=1, \text{ here}; \ \sigma = 1(x), 2(y))$$

$$(1) \ m\ddot{x} = -\lambda \frac{x}{b} = Q_x$$

$$(2) \ m\ddot{y} + mg = \lambda = Q_y$$

$$(3) \ y - \frac{x^2}{2b} = 0 \qquad \text{(constraint)}$$

(c) $H = E = \dot{x}p_x + \dot{y}p_y - L = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) + mgy$ conserved

(d) Substitute for y: $y = \frac{x^2}{2b} \Rightarrow \dot{y} = \frac{x}{b}\dot{x}$

$$\Rightarrow E = \frac{1}{2} m\left(1+\frac{x^2}{b^2}\right)\dot{x}^2 + mg\frac{x^2}{2b} = mg\frac{x_0^2}{2b}$$

Hence,

$$\dot{x}^2 = gb\,\frac{x_0^2 - x^2}{b^2+x^2}$$

We also need

$$\ddot{y} = \frac{x}{b}\ddot{x} + \frac{1}{b}\dot{x}^2$$

Hence from (2)

$$\lambda = m\ddot{y} + mg = m\left(g + \frac{1}{b}x\ddot{x} + \frac{1}{b}\dot{x}^2\right)$$

and substituting into (1),

$$m\ddot{x} = -\frac{mx}{b}\left(g + \frac{1}{b}x\ddot{x} + \frac{1}{b}\dot{x}^2\right)$$

Subtracting, we obtain

$$m\frac{\dot{x}^2}{b} + mg = \lambda\left(1+\frac{x^2}{b^2}\right)$$

$$\Rightarrow \quad \lambda = m\,\frac{g+\frac{\dot{x}^2}{b}}{1+\frac{x^2}{b^2}}$$

So we need $\dot{x}$ as a function of x. Get this from energy conservation:

$$E = \frac{1}{2}m\left(1+\frac{x^2}{b^2}\right)\dot{x}^2 + mg\frac{x^2}{2b} = mg\frac{x_0^2}{2b}$$

$$\Rightarrow \quad \dot{x}^2 = gb\,\frac{x_0^2-x^2}{b^2+x^2}$$

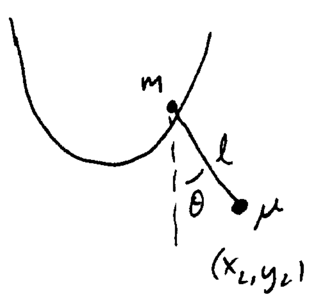
Thus,

$$Q_y = \lambda = mg\,\frac{1+\frac{x_0^2-x^2}{b^2+x^2}}{1+\frac{x^2}{b^2}} = mg\,b^2\,\frac{b^2+x_0^2}{(b^2+x^2)^2}$$

$$Q_x = -\frac{x}{b}\lambda = -mg\,b\,x\cdot\frac{b^2+x_0^2}{(b^2+x^2)^2}$$

Note $Q_y > 0$, $Q_x > 0$ if $x < 0$, $Q_x < 0$ if $x > 0$, as expected.

(e)



$$x_2 = x + l\sin\theta$$

$$y_2 = y - l\cos\theta$$

$$y = \frac{x^2}{2b}$$

$$\dot{x}_2 = \dot{x} + l\cos\theta\,\dot{\theta}$$

$$\dot{y}_2 = \dot{y} + l\sin\theta\,\dot{\theta} = \frac{x}{b}\dot{x} + l\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}\mu(\dot{x}_2^2+\dot{y}_2^2)$$
$$= \frac{1}{2}m\left(1+\frac{x^2}{b^2}\right)\dot{x}^2 + \frac{1}{2}\mu\left(1+\frac{x^2}{b^2}\right)\dot{x}^2 + \frac{1}{2}\mu l^2\dot{\theta}^2$$
$$+ \mu l\,\dot{x}\,\dot{\theta}\left(\cos\theta + \frac{x}{b}\sin\theta\right)$$

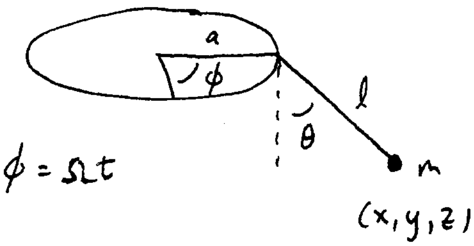
$$U = mgy + \mu g y_2$$
$$= mg\frac{x^2}{2b} + \mu g\left(\frac{x^2}{2b} - l\cos\theta\right)$$

$$L = \frac{1}{2}(m+\mu)\left(1+\frac{x^2}{b^2}\right)\dot{x}^2 + \frac{1}{2}\mu l^2\dot{\theta}^2 + \mu l\left(\cos\theta+\frac{x}{b}\sin\theta\right)\dot{x}\dot{\theta}$$
$$- (m+\mu)g\frac{x^2}{2b} + \mu g l\cos\theta$$

(f) $$p_x = \frac{\partial L}{\partial \dot{x}} = (m+\mu)\left(1+\frac{x^2}{b^2}\right)\dot{x} + \mu l\left(\cos\theta + \frac{x}{b}\sin\theta\right)\dot{\theta}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu l^2\dot{\theta} + \mu l\left(\cos\theta + \frac{x}{b}\sin\theta\right)\dot{x}$$

[2]



$\phi = \Omega t$

$$\begin{cases} x = (a + l\sin\theta)\cos\phi \\ y = (a + l\sin\theta)\sin\phi \\ z = -l\cos\theta \end{cases}$$

$$\dot{x} = -(a + l\sin\theta)\sin\phi\,\dot{\phi} + l\cos\theta\cos\phi\,\dot{\theta}$$
$$\dot{y} = (a + l\sin\theta)\cos\phi\,\dot{\phi} + l\cos\theta\sin\phi\,\dot{\theta}$$
$$\dot{z} = l\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\left(l^2\dot{\theta}^2 + (a + l\sin\theta)^2\dot{\phi}^2\right)$$
$$U = mgz$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}m\Omega^2(a + l\sin\theta)^2 + mgl\cos\theta$$

(b) $H = \dot{\theta}p_\theta - L \qquad ; \quad p_\theta = \dfrac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$

$$= \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}m\Omega^2(a + l\sin\theta)^2 - mgl\cos\theta$$

$$= \frac{p_\theta^2}{2ml^2} - \frac{1}{2}m\Omega^2(a + l\sin\theta)^2 - mgl\cos\theta$$

H is conserved since $-\dfrac{\partial L}{\partial t} = \dfrac{dH}{dt} = 0$. But $H \neq T + U$!

(c) $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) - \dfrac{\partial L}{\partial\theta} = 0$:

$$ml^2\ddot{\theta} = -mgl\sin\theta + m\Omega^2 l(a + l\sin\theta)\cos\theta$$

(d) $$\ddot{\theta} = -\frac{g}{l}\sin\theta + \Omega^2\left(\frac{a}{l} + \sin\theta\right)\cos\theta$$

$$= -\frac{\partial W_{eff}}{\partial\theta}$$

$$W_{eff}(\theta) = -\frac{g}{l}\cos\theta + \frac{1}{2}\Omega^2\left(\frac{a}{l} + \sin\theta\right)^2$$

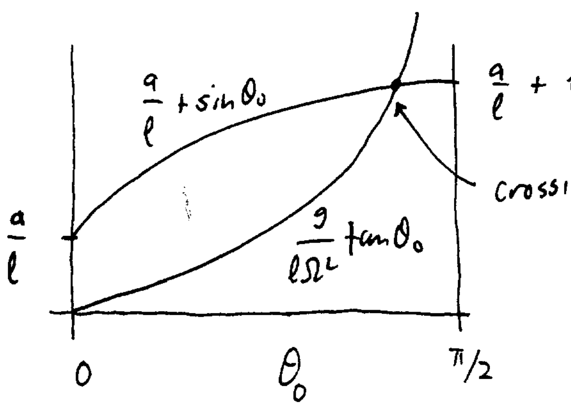
(e) Equilibrium $\Rightarrow$ $\ddot{\theta} = 0$ $\Rightarrow$ $\left.\frac{\partial W_{eff}}{\partial \theta}\right|_{\theta_0} = 0$:

$$\Omega^2 \left(\frac{a}{\ell} + \sin\theta_0 \right) \cos\theta_0 = \omega_0^2 \sin\theta_0$$

where $\omega_0 = \sqrt{\frac{g}{\ell}}$. This can also be written

$$\frac{a}{\ell} + \sin\theta_0 = \frac{g}{\ell\Omega^2} \tan\theta_0$$

Graphically, there is a unique solution $\theta_0 \in [0, \frac{\pi}{2}]$:



(f) Small oscillations about θ_0 : Let $\theta = \theta_0 + \epsilon$, in which case

$$\ddot{\epsilon} = -W''_{eff}(\theta_0)\,\epsilon + \cancel{\mathcal{O}(\epsilon^2)}$$

ϵ small

$\Rightarrow$ $\omega = \sqrt{W''_{eff}(\theta_0)}$ is small oscillation frequency

$$W''_{eff} = +\frac{g}{\ell}\cos\theta + \Omega^2 \frac{a}{\ell}\sin\theta - \Omega^2 \cos 2\theta$$

$\theta_0 = \frac{\pi}{4}$ $\Rightarrow$ $W''_{eff}(\theta_0) = \frac{1}{\sqrt{2}}\left(\frac{g}{\ell} + \Omega^2 \frac{a}{\ell} \right)$

But $\theta_0 = \frac{\pi}{4}$ $\Rightarrow$ $\frac{a}{\ell} + \frac{1}{\sqrt{2}} = \frac{g}{\ell\Omega^2}$

So,

$$\Omega^2 = \frac{g/\ell}{\frac{1}{\sqrt{2}} + \frac{a}{\ell}} = \frac{g}{\ell\sqrt{2}}$$

This gives

$$\omega^2 = W''_{eff}(\theta_0) = \frac{1}{\sqrt{2}}\frac{g}{\ell}\left(1+\frac{1}{2}\right) = \frac{3}{2\sqrt{2}}\frac{g}{\ell}$$

With $\omega_0 \equiv \sqrt{g/\ell}$, then,

$$\Omega = 2^{-1/4}\,\omega_0$$

$$\omega = \left(\frac{3}{4}\sqrt{2}\right)^{1/2}\omega_0$$