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Physics 110A: Problem Set #4

Due Monday, November 5 by 12:30 pm

Reading: MT chapters 6-7 ; lecture notes #4

[1] A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M which makes an angle α with the horizontal. Find the Euler-Lagrange equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

[2] Consider a particle moving in three dimensional space. The potential energy is $U(x, y, z) = U_1$ if $z < 0$ and $U(x, y, z) = U_2$ if $z \geq 0$. If a particle of mass m moving with speed v and at polar angle θ (*i.e.* the angle with respect to the $\hat{\mathbf{z}}$ axis) passes from the 'lower' half-space (*i.e.* the $z < 0$ region) into the 'upper' half space (*i.e.* the $z > 0$ region), show that in the latter region it moves with constant velocity v' and at polar angle θ'. Find v' and θ'. What is the optical analog to this problem?

[3] An enextensible massless string of length ℓ passes through a hole in a frictionless table. A point mass m on one end of the string moves on the table and a point mass m hangs from the other end.

(a) Write the Lagrangian for this system.

(b) Under what conditions will the hanging mass remain stationary?

(c) Starting from the situation in part (b), the hanging mass is pulled down slightly and released. State clearly what is conserved during this process.

(d) Compute the subsequent motion of the hanging mass.

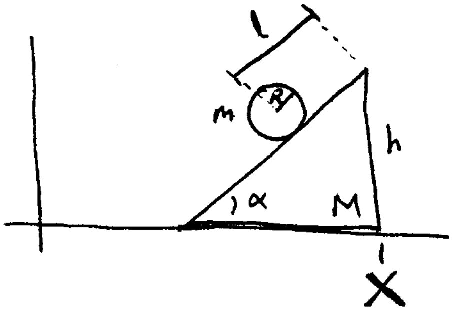
[4] The point of suspension of a pendulum of mass m is itself a mass, M, allowed to move in the horizontal direction. The mass M is connected to springs of force constant k on either side, providing a net restoring force $-2kx$ on the point of suspension.

(a) Use the generalized coordinates x (the displacement from equilibrium of the support mass M) and θ (the angular displacement of the pendulum from the vertical) to write the Lagrangian of the system.

(b) Solve for the small oscillations of this system.

[5] A uniform ladder of length L and mass M has one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest in a vertical plane perpendicular to the wall and makes an angle θ_0 with the horizontal. Make a convenient choice of generalized coordinates and find the Lagrangian. Derive the corresponding equations of motion. Prove that the ladder leaves the wall when its upper end has fallen to a height $\frac{2}{3} L \sin\theta_0$. Show how the subsequent motion can be reduced to explicit integrals. Does the ladder ever lose contact with the floor?

[1]



$l = R\phi$; ϕ = angle hoop rolls through

$X_{cm} = X - l\cos\alpha$

$y_{cm} = h + R - l\sin\alpha$

$\dot{x}_{cm} = \dot{X} - R\cos\alpha\,\dot{\phi}$

$\dot{y}_{cm} = -R\sin\alpha\,\dot{\phi}$

$$T = \tfrac{1}{2}m(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \tfrac{1}{2}I\dot{\phi}^2$$

$$= \tfrac{1}{2}mR^2\dot{\phi}^2 - mR\cos\alpha\,\dot{X}\dot{\phi} + \tfrac{1}{2}m\dot{X}^2 + \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}I\dot{\phi}^2$$

$$U = mgy_{cm} = -mgR\phi\sin\alpha + U_0 \leftarrow U_0 = mg(h+R) = \text{constant}$$

(i) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = \frac{\partial L}{\partial X} \Rightarrow (M+m)\ddot{X} - mR\cos\alpha\,\ddot{\phi} = 0$

(ii) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \Rightarrow (I+mR^2)\ddot{\phi} - mR\cos\alpha\,\ddot{X} = mgR\sin\alpha$

Integrals of motion : $p_x = \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} - mR\cos\alpha\,\dot{\phi}$

$$H = T + U = \tfrac{1}{2}(I+mR^2)\dot{\phi}^2 + \tfrac{1}{2}(M+m)\dot{X}^2 - mR\cos\alpha\,\dot{X}\dot{\phi} - mgR\sin\alpha\,\phi$$

Lagrangian : $L = T - U = \tfrac{1}{2}(I+mR^2)\dot{\phi}^2 + \tfrac{1}{2}(M+m)\dot{X}^2 - mR\cos\alpha\,\dot{X}\dot{\phi} + mgR\sin\alpha\,\phi$

Solution : $\ddot{X} = \frac{mR\cos\alpha}{M+m}\ddot{\phi}$

$$\left(I + mR^2 - \frac{m^2R^2\cos^2\alpha}{M+m}\right)\ddot{\phi} = mgR\sin\alpha$$

Note hoop $\Rightarrow$ $I = mR^2$ $\Rightarrow$

$$\left(2 - \frac{m\cos^2\alpha}{M+m}\right)\ddot{\phi} = \frac{g}{R}\sin\alpha$$

$$\phi(t) = \phi(0) + \dot{\phi}(0)t + \frac{1}{2}\frac{g/R}{2 - \frac{m\cos^2\alpha}{}}t^2$$

[2] We have energy conservation

$$E = \tfrac{1}{2} m v_1^2 + U_1 + \tfrac{1}{2} m v_2^2 + U_2$$

We also have conservation of transverse momentum (hence velocity), so

$$\left.\begin{aligned} v_1^2 &= v_\perp^2 + v_{1z}^2 \\ v_2^2 &= v_\perp^2 + v_{2z}^2 \end{aligned}\right\} \quad (\text{same } v_\perp)$$

Let us define $c^2 \equiv \frac{2E}{m}$. Then

$$E = E\,\frac{v_1^2}{c^2} + U_1 \Rightarrow v_1 = \frac{c}{\sqrt{1 - \frac{U_1}{E}}}$$

$$E = E\,\frac{v_2^2}{c^2} + U_2 \Rightarrow v_2 = \frac{c}{\sqrt{1 - \frac{U_2}{E}}}$$

Furthermore,

$$\cos^2\theta_1 = \frac{v_{1z}^2}{v_1^2} = 1 - \frac{v_\perp^2}{v_1^2} \Rightarrow \sin^2\theta_1 = \frac{\frac{1}{2} m v_\perp^2}{E - U_1}$$

$$\cos^2\theta_2 = \frac{v_{2z}^2}{v_2^2} = 1 - \frac{v_\perp^2}{v_2^2} \Rightarrow \sin^2\theta_2 = \frac{\frac{1}{2} m v_\perp^2}{E - U_2}$$

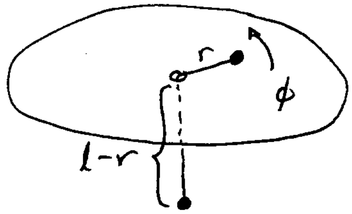
$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{\sqrt{1 - \frac{U_1}{E}}}{\sqrt{1 - \frac{U_2}{E}}}$$

This is simply geometrical optics (Snell's law) in another guise:

$$n = \sqrt{1 - \frac{U}{E}}, \quad v = \frac{c}{n}$$

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$$

[3]



$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\dot{r}^2$$

$$U = mg(r-l)$$

$$L = T - U = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr \quad \text{(ignore constant)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} = mr\dot{\phi}^2 - mg = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\big(\underbrace{mr^2\dot{\phi}}_{P_\phi}\big) = 0 = \frac{\partial L}{\partial \phi}$$

Thus, $p_\phi = mr^2\dot{\phi}$ is conserved. Substituting $\dot{\phi} = \frac{p_\phi}{mr^2}$, we have

$$\ddot{r} = -g + \frac{p_\phi^2}{m^2r^3}$$

Stationary condition: $\ddot{r} = 0 \Rightarrow r = r_0$ with $r_0^3 = \frac{p_\phi^2}{m^2 g} = r_0^4 \frac{\dot{\phi}^2}{g}$

I.e. $\dot{\phi}^2 r_0 = g$, and the centripetal acceleration balances the acceleration due to gravity. Now suppose the hanging mass is pulled (or pushed) slightly in the vertical direction. This doesn't introduce any torque, hence p_ϕ remains conserved. We write $r = r_0 + \delta r$ and linearize the equations of motion:

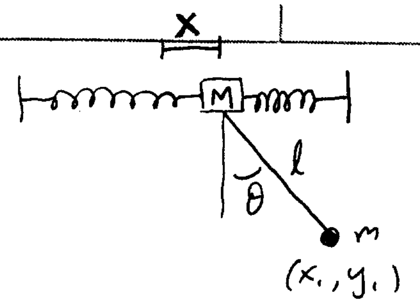
$$\delta\ddot{r} = \underbrace{-g + \frac{p_\phi^2}{m^2 r_0^3}}_{=0} - \frac{3p_\phi^2}{m^2 r_0^4}\,\delta r + \mathcal{O}(\delta r^2)$$

Thus,

$$\delta\ddot{r} = -\Omega^2\,\delta r \Rightarrow \delta r(t) = A\cos(\Omega t + \eta)$$

constants

$$\Omega = \left(\frac{3p_\phi^2}{m^2 r_0^4}\right)^{1/2} = \sqrt{3}\cdot\left(\frac{mg^2}{p_\phi}\right)^{1/3}$$

[4] 

x = displacement of M from equilibrium

$x_1 = x + l\sin\theta \Rightarrow \dot{x}_1 = \dot{x} + l\cos\theta\,\dot{\theta}$

$y_1 = -l\cos\theta \Rightarrow \dot{y}_1 = l\sin\theta\,\dot{\theta}$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{x}\,\dot{\theta}$$

$$U = kx^2 - mgl\cos\theta$$

$$L = T - U = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{x}\dot{\theta} + mgl\cos\theta - kx^2$$

Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = -2kx = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + ml\cos\theta\,\ddot{x} \cancel{- ml\sin\theta\,\dot{x}\dot{\theta}} = \cancel{-ml\sin\theta\,\dot{x}\dot{\theta}} - mgl\sin\theta = \frac{\partial L}{\partial \theta}$$

Thus,

$$\left(1 + \frac{m}{M}\right)\ddot{x} + \frac{m}{M}l\cos\theta\,\ddot{\theta} - \frac{m}{M}l\sin\theta\,\dot{\theta}^2 = -\frac{2k}{M}x$$

$$l\ddot{\theta} + \cos\theta\,\ddot{x} = -g\sin\theta$$

Linearization: assume $\theta \simeq 0$, $x \simeq 0$.

$$\left(1 + \frac{m}{M}\right)\ddot{x} + \frac{m}{M}l\ddot{\theta} + \frac{2k}{M}x = 0$$

$$\ddot{x} + l\ddot{\theta} + g\theta = 0$$

Try a solution

$$\begin{pmatrix} x(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix} e^{-i\omega t}$$

Then

$$\begin{pmatrix} \frac{2k}{M} - \left(1+\frac{m}{M}\right)\omega^2 & -\frac{m}{M} l \omega^2 \\ -\omega^2 & g - l\omega^2 \end{pmatrix} \begin{pmatrix} x_0 \\ \theta_0 \end{pmatrix} = 0$$

For a nontrivial solution, the determinant must vanish:

$$\left[\frac{2k}{M} - \left(1+\frac{m}{M}\right)\omega^2\right]\left[g - l\omega^2\right] - \frac{m}{M} l \omega^4 = 0$$

$$\Rightarrow \quad l\omega^4 - \left[\left(1+\frac{m}{M}\right)g + \frac{2k}{M} l\right]\omega^2 + \frac{2k}{M} g = 0$$

Define $\quad \nu^2 \equiv \frac{g}{l}, \quad \Omega^2 \equiv \frac{2k}{M}, \quad \lambda \equiv \frac{m}{M}$:

$$\omega^4 - \left[(1+\lambda)\nu^2 + \Omega^2\right]\omega^2 + \Omega^2\nu^2 = 0$$

$$\omega^2 = \frac{(1+\lambda)\nu^2 + \Omega^2 \pm \sqrt{\left[(1+\lambda)\nu^2 + \Omega^2\right]^2 - 4\Omega^2\nu^2}}{2}$$

$$\omega_\pm^2 = \frac{1}{2}\left[(1+\lambda)\nu^2 + \Omega^2\right] \pm \frac{1}{2}\sqrt{\lambda^2\nu^4 + 2\lambda\nu^2(\nu^2+\Omega^2) + (\nu^2-\Omega^2)^2}$$

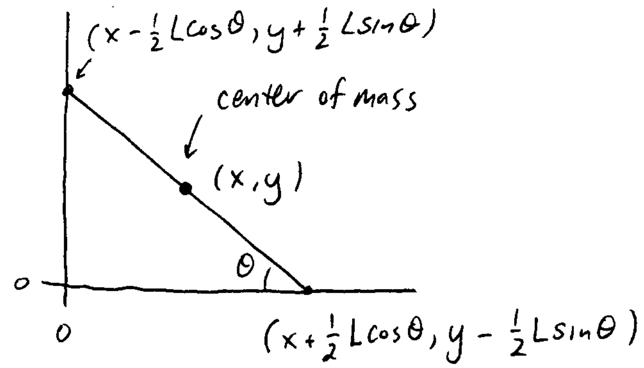
Note :

$$\lambda = 0 \Rightarrow \omega_\pm^2 = \frac{1}{2}(\nu^2+\Omega^2) \pm \frac{1}{2}(\nu^2-\Omega^2) = \nu^2, \Omega^2$$

$$\Omega = 0 \Rightarrow \omega_\pm^2 = \frac{1}{2}(1+\lambda)\nu^2 \pm \frac{1}{2}(1+\lambda)\nu^2 = 0, (1+\lambda)\nu^2$$

$$\nu = 0 \Rightarrow \omega_\pm^2 = \frac{1}{2}\Omega^2 \pm \frac{1}{2}\Omega^2 = 0, \Omega^2$$

[5]



Constraints: $G_1(x,y,\theta) = x - \frac{1}{2}L\cos\theta$ (left edge on wall)

$G_2(x,y,\theta) = y - \frac{1}{2}L\sin\theta$ (bottom on floor)

Kinetic energy: $T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$ $\qquad I_{rod} = \frac{1}{12}ML^2$

Potential energy: $U = Mgy$ $\qquad$ Lagrangian: $L = T - U$

Euler-Lagrange equations from $L = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - Mgy$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \lambda_1 \frac{\partial G_1}{\partial q_\sigma} + \lambda_2 \frac{\partial G_2}{\partial q_\sigma} \qquad q_\sigma : x, y, \theta$$

(i) $M\ddot{x} = \lambda_1 = Q_x$

(ii) $M\ddot{y} + Mg = \lambda_2 = Q_y$

(iii) $I\ddot{\theta} = \frac{1}{2}L(+\lambda_1\sin\theta - \lambda_2\cos\theta) = Q_\theta$

Use the constraint equations $G_1 = 0$, $G_2 = 0$ to obtain x, y in terms of θ:

$$x = \frac{1}{2}L\cos\theta \Rightarrow \dot{x} = -\frac{1}{2}L\sin\theta\,\dot{\theta} \Rightarrow \ddot{x} = -\frac{1}{2}L\cos\theta\,\dot{\theta}^2 - \frac{1}{2}L\sin\theta\,\ddot{\theta}$$

$$y = \frac{1}{2}L\sin\theta \Rightarrow \dot{y} = \frac{1}{2}L\cos\theta\,\dot{\theta} \Rightarrow \ddot{y} = -\frac{1}{2}L\sin\theta\,\dot{\theta}^2 + \frac{1}{2}L\cos\theta\,\ddot{\theta}$$

Thus, we obtain the forces of constraint $\lambda_1 = Q_x$ and $\lambda_2 = Q_y$

$$\lambda_1 = -\frac{1}{2}ML\left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right)$$

$$\lambda_2 = \frac{1}{2}ML\left(-\sin\theta\,\dot{\theta}^2 + \cos\theta\,\ddot{\theta}\right) + Mg$$

Substitute these into (iii) to obtain

$$\begin{aligned} I\ddot{\theta} &= +\frac{1}{2}L\sin\theta\,\lambda_1 - \frac{1}{2}L\cos\theta\,\lambda_2 \\ &= -\frac{1}{4}ML^2\left(\cancel{\sin\theta\cos\theta\,\dot{\theta}^2} + \sin^2\theta\,\ddot{\theta} - \cancel{\sin\theta\cos\theta\,\dot{\theta}^2} + \cos^2\theta\,\ddot{\theta}\right) - \frac{1}{2}MgL\cos\theta \\ &= -\frac{1}{4}ML^2\ddot{\theta} - \frac{1}{2}MgL\cos\theta \end{aligned}$$

Define $I_0 = \frac{1}{4}ML^2$. This is the largest possible value of I, corresponding to placing all the mass at the endpoints. We therefore obtain

$$\left(1 + \frac{I}{I_0}\right)\ddot{\theta} = -2\omega_0^2\cos\theta \qquad \omega_0^2 \equiv \frac{g}{L}$$

We integrate this once to obtain

$$\frac{1}{2}\left(1 + \frac{I}{I_0}\right)\dot{\theta}^2 + 2\omega_0^2\sin\theta = 2\omega_0^2\sin\theta_0$$

Thus,

$$\dot{\theta}^2 = \frac{4\omega_0^2\left(\sin\theta_0 - \sin\theta\right)}{1 + \frac{I}{I_0}}$$

$$\ddot{\theta} = -\frac{2\omega_0^2\cos\theta}{1 + \frac{I}{I_0}}$$

We can now obtain $\lambda_1(\theta)$ and $\lambda_2(\theta)$ in terms of θ alone.

Substituting $y = \frac{1}{2} L \sin\theta \Rightarrow \ddot{y} = -\frac{1}{2} L \sin\theta\, \dot{\theta}^2 + \frac{1}{2} L \cos\theta\, \ddot{\theta}$

we have

$$\lambda = Mg - \tfrac{1}{2} ML \sin\theta\, \dot{\theta}^2 + \tfrac{1}{2} ML \cos\theta\, \ddot{\theta}$$

$$I\ddot{\theta} = -\tfrac{1}{2} MgL \cos\theta + \tfrac{1}{4} ML^2 \sin\theta \cos\theta\, \dot{\theta}^2 - \tfrac{1}{4} ML^2 \cos^2\theta\, \ddot{\theta}$$

$$I\ddot{\theta}\dot{\theta} = -\tfrac{1}{2} MgL \cos\theta\, \dot{\theta} + \tfrac{1}{4} ML^2 \left(\sin\theta \cos\theta\, \dot{\theta}^3 - \cos^2\theta\, \ddot{\theta}\dot{\theta}\right)$$

$$\frac{d}{dt}\left(\tfrac{1}{2} I \dot{\theta}^2\right) = -\frac{d}{dt}\left(\tfrac{1}{2} MgL \sin\theta\right) - \tfrac{1}{4} ML^2 \frac{d}{dt}\left(\tfrac{1}{2}\cos^2\theta\, \dot{\theta}^2\right)$$

$$\Rightarrow \quad \tfrac{1}{2}\left(I + I_0 \cos^2\theta\right)\dot{\theta}^2 + \tfrac{1}{2} MgL \sin\theta = C \qquad \text{(a constant)}$$

By continuity with our original solution, we know that the initial conditions for this second phase of the motion are

$$\theta = \sin^{-1}\left(\tfrac{2}{3}\sin\theta_0\right) \iff \sin\theta = \tfrac{2}{3}\sin\theta_0$$

$$\dot{\theta}^2 = \frac{4}{3} \frac{\omega_0^2}{1 + \frac{I}{I_0}} \sin\theta_0$$

Thus,

$$C = \tfrac{1}{2}\left(I + I_0 - \tfrac{4}{9} I_0 \sin^2\theta_0\right) \cdot \frac{4}{3} \frac{\omega_0^2}{1 + \frac{I}{I_0}} \sin\theta_0 + \tfrac{1}{3} MgL \sin\theta_0$$

$$= \left[\frac{2}{3} \cdot \frac{1 - \frac{4}{9}\sin^2\theta_0 + \frac{I}{I_0}}{1 + \frac{I}{I_0}} \sin\theta_0 + \frac{4}{3}\sin\theta_0\right] I_0 \omega_0^2$$

$$= 2 I_0 \omega_0^2 \cdot \frac{1 + \frac{I}{I_0} - \frac{4}{27}\sin^2\theta_0}{1 + \frac{I}{I_0}} \sin\theta_0$$

We then have to integrate

$$\frac{d\theta}{dt} = -\sqrt{\frac{C - 2 I_0 \omega_0^2 \sin\theta}{\frac{1}{2}\left(I + I_0 \cos^2\theta\right)}}$$

$$\lambda_1 = -\tfrac{1}{2}ML\cos\theta\,\dot\theta^2 - \tfrac{1}{2}ML\sin\theta\,\ddot\theta$$

$$= -\tfrac{1}{2}ML\cdot\frac{2\omega_0^2}{1+\frac{I}{I_0}}\left\{2(\sin\theta_0-\sin\theta)\cos\theta - \sin\theta\cos\theta\right\}$$

$$= \frac{Mg}{1+\frac{I}{I_0}}\left(3\sin\theta - 2\sin\theta_0\right)\cos\theta$$

* note this is independent of the value I/I_0

Thus we see $\lambda_1 = 0$ when $\sin\theta = \frac{2}{3}\sin\theta_0$ *, i.e. the point of contact with the wall has fallen to a height $\frac{2}{3}L\sin\theta_0$. What about λ_2 ?

$$\lambda_2 = Mg - \tfrac{1}{2}ML\sin\theta\,\dot\theta^2 + \tfrac{1}{2}ML\cos\theta\,\ddot\theta$$

$$= Mg - \tfrac{1}{2}ML\frac{4\omega_0^2}{1+\frac{I}{I_0}}(\sin\theta_0-\sin\theta)\sin\theta - \tfrac{1}{2}ML\frac{2\omega_0^2\cos\theta}{1+\frac{I}{I_0}}\cos\theta$$

$$= \frac{Mg}{1+\frac{I}{I_0}}\left\{1+\frac{I}{I_0} - 2\sin\theta_0\sin\theta + 2\sin^2\theta - \cos^2\theta\right\}$$

$$= \frac{Mg}{1+\frac{I}{I_0}}\cdot\left\{+\frac{I}{I_0} + (3\sin\theta - 2\sin\theta_0)\sin\theta\right\}$$

We see $\lambda_2(\theta)$ is always positive, hence the normal force never vanishes and the ladder never leaves the floor.

What happens after the ladder detaches from the wall? At that point the first constraint is no longer applicable and we have

$$M\ddot x = 0 \qquad \text{(conservation of momentum along x-axis)}$$

$$M\ddot y + Mg = \lambda$$

$$I\ddot\theta = -\tfrac{L}{2}\lambda\cos\theta$$

$$G = y - \tfrac{1}{2}L\sin\theta = 0 \qquad \text{(constraint at floor)}$$