Physics 110A: Problem Set #3

Due Monday, October 29 by 12:30 pm

Reading: MT chapter 5 ; lecture notes #4

[1] Find the differential equation which results from extremization of the following functionals. Specify whatever boundary conditions are required.

$$(a) \qquad \mathcal{F}[y(x)] = \int_a^b dx \left[\tfrac{1}{2}(y')^2 - e^{-yy'} \right]$$

$$(b) \qquad \mathcal{F}[y(x)] = \int_a^b dx \left[\tfrac{1}{2}\frac{(y')^2}{1+y^2} + \tfrac{1}{2}(y'')^2\, e^{-y} \right]$$

[2] A cylinder has height L and radius a. Maximize its volume subject to the constraint $L^4 + a^4 = 1$. (Find a, L, and V.)

[3] The speed of an object moving along a plane depends on the object's distance from a single point (the origin): $v = v(r)$. Construct the time functional $T[\phi(r)]$, where $\phi(r)$ is the object's path represented by azimuthal angle as a function of radius. Find the equation of the extremizing path connecting the points $(r = a, \phi = 0)$ and $(r = b, \phi = \frac{1}{2}\pi)$ for each of the following $v(r)$:

(a) $v(r) = v_0$ (v_0 a constant)

(b) $v(r) = r/\tau$ (τ a constant).

[4] A bumpy cylinder is defined by $\rho(z) = a + b\sin(z/\lambda)$, where $0 < b < a$. What differential equation must be solved to obtain geodesics $\phi(z)$ on such a surface?

[1] (a) $L(y,y') = \frac{1}{2}(y')^2 - e^{-yy'}$

$$\frac{\partial L}{\partial y} = y' e^{-yy'} \quad ; \quad \frac{\partial L}{\partial y'} = y' + y e^{-yy'}$$

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0 = \cancel{y'e^{-yy'}} - y'' - \cancel{y'e^{-yy'}} + y(y'^2 + yy'')e^{-yy'}$$

$$\Rightarrow \quad y''\left(y^2 e^{-yy'} - 1\right) + y e^{-yy'} y'^2 = 0$$

This can be integrated once :

$$E = y' \frac{\partial L}{\partial y'} - L = \frac{1}{2}(y')^2 + (yy'+1)\,e^{-yy'}$$

is conserved. Boundary conditions : $\delta y(a) = \delta y(b) = 0$.

(b) $L(y,y') = \frac{1}{2}\frac{(y')^2}{1+y^2} + \frac{1}{2}e^{-y}(y'')^2$

$$\frac{\partial L}{\partial y} = -\frac{y(y')^2}{(1+y^2)^2} - \frac{1}{2}e^{-y}(y'')^2$$

$$\frac{\partial L}{\partial y'} = \frac{y'}{1+y^2} \quad ; \quad \frac{\partial L}{\partial y''} = e^{-y}y''$$

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial L}{\partial y''}\right) = 0$$

$$0 = -\frac{y(y')^2}{(1+y^2)^2} - \frac{1}{2}e^{-y}(y'')^2 - \frac{y''}{1+y^2} + \frac{2y(y')^2}{(1+y^2)^2}$$

$$+ e^{-y}y'''' - 2y'y'''e^{-y} - (y'')^2 e^{-y} + (y')^2 y'' e^{-y}$$

Boundary conditions : $\delta y(a) = \delta y(b) = \delta y'(a) = \delta y'(b) = 0$.

[2] $V = \pi a^2 L \quad ; \quad L^4 + a^4 = 1$

$$V^* = \pi a^2 L + \lambda (L^4 + a^4 - 1)$$

$$\frac{\partial V^*}{\partial a} = 0 = 2\pi a L + 4\lambda a^3 \Rightarrow \lambda = -\frac{\pi}{2} L a^{-2}$$

$$\frac{\partial V^*}{\partial L} = 0 = \pi a^2 + 4\lambda L^3 \Rightarrow a^4 = 2L^4$$

$$\frac{\partial V^*}{\partial \lambda} = 0 = L^4 + a^4 - 1 \Rightarrow L = 3^{-1/4}, \quad a = \left(\frac{2}{3}\right)^{1/4}$$

$$\Rightarrow V = \frac{\sqrt{2}\,\pi}{3^{3/4}}$$

[3] $$T[\phi(r)] = \int_a^b dr \, \frac{\sqrt{1 + r^2(\phi')^2}}{v(r)} = \int \frac{ds}{v}$$

$$L(\phi, \phi', r) = \frac{\sqrt{1 + r^2(\phi')^2}}{v(r)} \Rightarrow \frac{\partial L}{\partial \phi'} = P = \text{constant}$$

$$P = \frac{1}{v(r)} \frac{r^2 \phi'}{\sqrt{1 + r^2(\phi')^2}} \Rightarrow \frac{d\phi}{dr} = \pm \frac{P v(r)}{r\sqrt{r^2 - P^2 v^2(r)}}$$

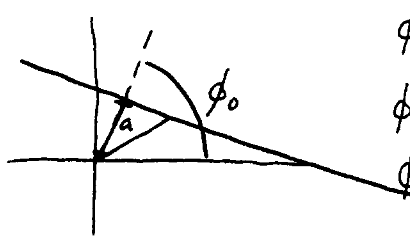
(a) $v(r) = v_0 \quad ; \quad P v_0 \equiv d$

$$\frac{d\phi}{dr} = \pm \frac{P v_0}{r\sqrt{r^2 - P^2 v_0^2}} \equiv \pm \frac{d}{r\sqrt{r^2 - d^2}}$$

Substitute $r = d\sec\theta \Rightarrow r^2 - d^2 = d^2 \tan^2\theta, \quad dr = +d \dfrac{\sin\theta}{\cos^2\theta} d\theta$

$$d\phi = \pm d^2 \frac{\sin\theta}{\cos^2\theta} d\theta \cdot \frac{1}{d \tan\theta \sec\theta} = \pm d\theta \quad ✓$$

$$\Rightarrow r(\phi) = \frac{d}{\cos(\phi - \phi_0)} \Rightarrow \text{straight line !}$$



$$\phi(r) = \phi_0 + \cos^{-1}\left(\frac{d}{r}\right)$$

$$\phi(a) = \phi_0 + \cos^{-1}\left(\frac{d}{a}\right)$$

$$\phi(b) = \phi_0 + \cos^{-1}\left(\frac{d}{b}\right)$$

(b) $v(r) = \frac{r}{\tau} \Rightarrow \frac{d\phi}{dr} = \pm \frac{Pr/\tau}{r^2\sqrt{1-P^2/\tau^2}}$

Let $\lambda \equiv P/\tau$ (dimensionless) $\rightarrow$

$$\phi'(r) = \pm \frac{\lambda}{\sqrt{1-\lambda^2}} \cdot \frac{1}{r}$$

$$\phi(r) = \phi_0 \pm \frac{\lambda}{\sqrt{1-\lambda^2}} \ln\left(\frac{r}{a}\right)$$

I.e. a logarithmic spiral (opening clockwise (−) or counterclockwise (+))

$$\phi(a) = \phi_0 \pm \frac{\lambda}{\sqrt{1-\lambda^2}} \ln(1) = \phi_0$$

$$\phi(b) = \phi(a) \pm \frac{\lambda}{\sqrt{1-\lambda^2}} \ln\left(\frac{b}{a}\right)$$

(solve for λ)

[4] $\rho(z) = a + b\sin(z/\lambda)$

$$ds^2 = (dz)^2 + (d\rho)^2 + \rho^2(d\phi)^2$$

$$= dz\sqrt{1 + \frac{b^2}{\lambda^2}\cos^2\left(\frac{z}{\lambda}\right) + \left[a+b\sin\left(\frac{z}{\lambda}\right)\right]\left(\frac{d\phi}{dz}\right)^2}$$

$$D[\phi(z)] = \int ds = \int_a^b dz\,\sqrt{1+(\rho')^2 + \rho^2(\phi')^2}$$

$$\frac{\partial L}{\partial \phi'} = P = \frac{\rho^2\phi'}{\sqrt{1+(\rho')^2+\rho^2(\phi')^2}} = \text{constant}$$

$$\rho^2\left(\frac{\rho^2}{P^2} - 1\right)\phi'^2 = 1 + (\rho')^2$$

$$\frac{d\phi}{dz} = \pm \frac{\sqrt{1+\left(\frac{d\rho}{dz}\right)^2}}{\rho(z)\sqrt{\frac{\rho^2(z)}{P^2} - 1}} = \frac{\pm\sqrt{1 + \frac{b^2}{\lambda^2}\cos^2\left(\frac{z}{\lambda}\right)}}{\left(a+b\sin\left(\frac{z}{\lambda}\right)\right)\sqrt{\left(\frac{a+b\sin\left(\frac{z}{\lambda}\right)}{P}\right)^2 - 1}}$$