PHYSICS 110A, SOLUTION SET #2

[1] $m\ddot{x} = -Kx \Rightarrow T = \frac{1}{2}m\dot{x}^2$, $U = \frac{1}{2}Kx^2$; $E = T + U$

$x(t) = a\sin(\omega t + \varphi)$; $\omega = \left(\frac{K}{m}\right)^{1/2}$; $T = \frac{2\pi}{\omega}$ = period

• time averages:

$$\langle T\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\ \frac{1}{2}ma^2\omega^2\cos^2(\omega t+\varphi) = \frac{1}{4}ma^2\omega^2 = \frac{1}{2}E$$

$$\langle U\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\ \frac{1}{2}Ka^2\sin^2(\omega t+\varphi) = \frac{1}{4}Ka^2 = \frac{1}{2}E$$

• space averages: use $\dot{x}^2 = \omega^2(a^2-x^2)$

$$\bar{T} = \frac{1}{2}m\omega^2\cdot\frac{1}{a}\int_0^a dx\,(a^2-x^2) = \frac{1}{3}m\omega^2a^2 = \frac{2}{3}E$$

$$\bar{U} = \frac{1}{2}Ka^2\cdot\frac{1}{a}\int_0^a dx\ x^2 = \frac{1}{6}Ka^2 = \frac{1}{3}E$$

Thus, $\langle T\rangle = \langle U\rangle = \frac{1}{2}E$; $\bar{T} = 2\bar{U} = \frac{2}{3}E$.

[2] $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0\cos\Omega t$

$\Rightarrow x(t) = Rf_0\cos(\Omega t - \delta)$; $R = \left[(\omega_0^2-\Omega^2)^2 + 4\beta^2\Omega^2\right]^{-1/2}$

(assume homogeneous sol^n fully damped to zero) $\quad \delta = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2-\Omega^2}\right)$

Energy lost by external source per cycle:

$$\Delta\mathcal{E} = \int dx\ f = \int dt\ \dot{x}\,f$$

$$= -\int_0^{2\pi/\Omega} dt\ R\Omega f_0^2\sin(\Omega t-\delta)\cos\Omega t = \frac{\pi}{\Omega}R\Omega f_0^2\sin\delta$$

$$= 2\pi\beta\Omega R^2 f_0^2 \quad \text{since } \sin\delta = 2\beta\Omega\cdot R$$

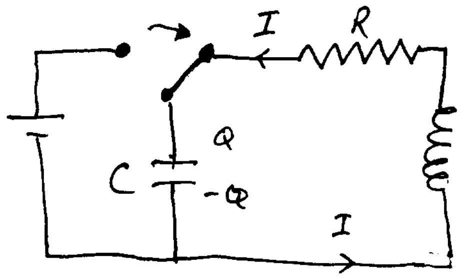
$$\langle\mathcal{E}\rangle = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} dt\ \frac{1}{2}(\dot{x}^2 + \omega_0^2x^2) = \frac{1}{4}(\Omega^2+\omega_0^2)R^2f_0^2$$

$$\Rightarrow 2\pi\frac{\langle\mathcal{E}\rangle}{\Delta\mathcal{E}} = \frac{2\pi\cdot\frac{1}{4}\cdot(\Omega^2+\omega_0^2)\cancel{R^2f_0^2}}{2\pi\beta\Omega\cancel{R^2f_0^2}} = \frac{\Omega^2+\omega_0^2}{4\beta\Omega}$$

$Q = \frac{\Omega_R}{2\beta}$; $\Omega_R = \sqrt{\omega_0^2 - 2\beta^2}$, so for $\Omega \approx \Omega_R$ $\quad \frac{2\pi\langle\mathcal{E}\rangle}{\Delta\mathcal{E}} \approx Q$

[3]

$V = 10\text{ V}$



$L = 10\text{ mH}$

$R = 100\,\Omega$

$\Omega_R = 10\text{ kHz}$

Recall $\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = 0$ for the R-L-C circuit

$\Rightarrow \beta = \frac{R}{2L}$, $\omega_0^2 = \frac{1}{LC}$. Initially (at $t = 0^+$),

we have $Q(0) = CV$, $\dot{Q}(0) = 0$.

$$Q(t) = C_+ e^{-i\omega_+ t} + C_- e^{-i\omega_- t}$$

$$\omega_\pm = -i\frac{R}{2L} \pm \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

(a) $$\Omega_R = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}} = 10^4\,s^{-1}$$

$$\frac{1}{C} = \frac{R^2}{2L} + 10^8\,s^{-2}\cdot L$$

MKS units $\Rightarrow$

$$C = \frac{1\text{ F}}{10^8 \cdot 10^{-2} + \frac{10^4}{2\cdot 10^{-2}}} = \frac{2}{3}\times 10^{-6}\text{ F} = \frac{2}{3}\mu\text{F}$$

(b) $$Q(t) = C_+ e^{-i\omega_+ t} + C_- e^{-i\omega_- t}$$

$$\dot{Q}(t) = -i\omega_+ C_+ e^{-i\omega_+ t} - i\omega_- C_- e^{-i\omega_- t}$$

$$Q(0) = CV = C_+ + C_-$$

$$\dot{Q}(0) = 0 = -i\omega_+ C_+ - i\omega_- C_- \Rightarrow C_- = -\frac{\omega_+}{\omega_-}C_+$$

$$CV = C_+\left(1 - \frac{\omega_+}{\omega_-}\right) \Rightarrow C_+ = -\frac{\omega_- CV}{\omega_+ - \omega_-}$$

$$C_- = \frac{\omega_+ CV}{\omega_+ - \omega_-}$$

Thus,

$$Q(t) = e^{-\beta t}\left\{\cos\nu t + \frac{\beta}{\nu}\sin\nu t\right\}\cdot CV$$

with $\beta = \frac{R}{2L} = 5\times10^{3}\,s^{-1} = 5\,kHz$, $\quad \nu = \sqrt{\omega_0^2-\beta^2} \Rightarrow$

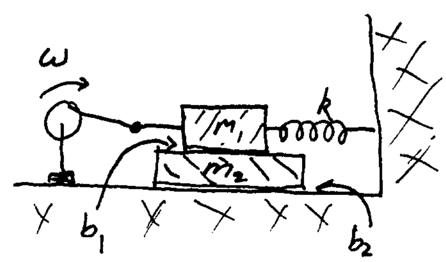
$$\nu = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{\frac{3}{2}\times10^{8} - \frac{1}{4}\times10^{8}}\ s^{-1} = \frac{\sqrt{5}}{2}\times10^{4}\,s^{-1}$$
$$= \sqrt{5}\cdot\beta$$

$$\dot{Q}(t) = -\beta e^{-\beta t}\left\{\cos\nu t + \frac{\beta}{\nu}\sin\nu t\right\}\cdot CV$$
$$+ e^{-\beta t}\left\{-\nu\sin\nu t + \beta\cos\nu t\right\}\cdot CV$$
$$= -e^{-\beta t}\cdot\left(\nu + \frac{\beta^2}{\nu}\right)\sin\nu t \cdot CV$$

$$CV = 10\,V\cdot\frac{2}{3}\,\mu F = 6\tfrac{2}{3}\,\mu C$$

$$\dot{Q} = -\frac{6}{\sqrt{5}}\cdot 5\,kHz\cdot e^{-5\tau}\cdot\sin(5\sqrt{5}\,\tau)\cdot\frac{20}{3}\,\mu C$$
$$= -40\sqrt{5}\,e^{-5\tau}\sin(5\sqrt{5}\,\tau)\ mA \qquad (\tau \text{ in msec})$$

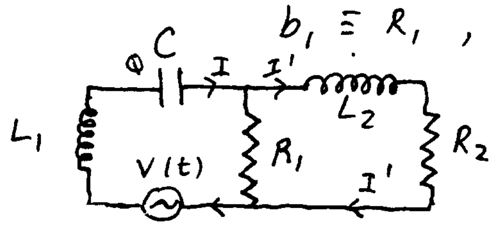
[4] Let x_1 = displacement of top block relative to right wall, x_2 displacement of bottom block.



Mechanical system:

$$m_1\ddot{x}_1 = -kx_1 - b_1(\dot{x}_1 - \dot{x}_2) + f(t)$$
$$m_2\ddot{x}_2 = -b_2\dot{x}_2 - b_1(\dot{x}_2 - \dot{x}_1)$$

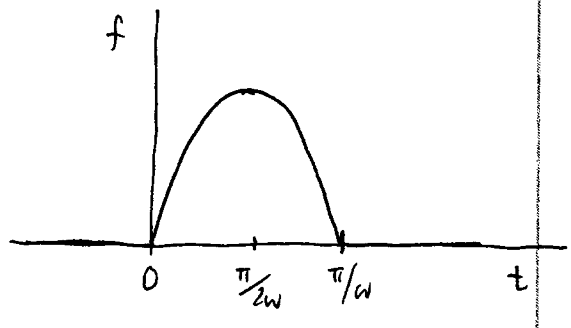
Circuit analogy: $m_1 \equiv L_1$, $m_2 \equiv L_2$, $x_1 \equiv Q$, $\dot{x}_1 = \dot{Q} = I$, $\dot{x}_2 \equiv I'$, $b_1 \equiv R_1$, $b_2 \equiv R_2$, $f(t) \equiv V(t)$; $k \equiv C$



$$\boxed{\begin{aligned}&\frac{Q}{C} + L_1\ddot{Q} + R_1(\dot{Q} - I') = V(t)\\ &L_2\dot{I}' + R_2 I' + R_1(I' - \dot{Q}) = 0\end{aligned}}$$ ✓

[5] $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = a \sin\omega t \;\Theta(t)\,\Theta(\pi - \omega t)$

Green's function:



$$G(s) = \nu^{-1} e^{-\beta s} \sin(\nu s)\;\Theta(s)$$

with $\nu = \sqrt{\omega_0^2 - \beta^2}$. Thus,

$$x(t) = \int_{-\infty}^{\infty} dt'\, G(t-t')\, f(t')$$

$$= \int_0^{\pi/\omega} dt'\, G(t-t')\; a\sin(\omega t')$$

$$= a\nu^{-1}\int_0^{\pi/\omega} dt'\; e^{-\beta(t-t')}\;\frac{e^{i\nu(t-t')} - e^{-i\nu(t-t')}}{2i}\;\frac{e^{i\omega t'} - e^{-i\omega t'}}{2i}$$

$$= \frac{a}{2\nu}\,\mathrm{Re}\int_0^{\pi/\omega} dt'\; e^{-\beta t} e^{\beta t'}\left\{ e^{-i\nu t} e^{i(\omega+\nu)t'} - e^{i\nu t} e^{i(\omega-\nu)t'}\right\}$$

$$= \frac{a}{2\nu}\, e^{-\beta t}\;\mathrm{Re}\int_0^{\pi/\omega} dt'\left\{ e^{-i\nu t} e^{(\beta+i\omega+i\nu)t'} - e^{i\nu t} e^{(\beta+i\omega-i\nu)t'}\right\}$$

$$= \frac{a}{2\nu}\, e^{-\beta t}\;\mathrm{Re}\left\{ \frac{e^{-i\nu t}}{\beta+i\omega+i\nu}\left(e^{(\beta+i\nu)\frac{\pi}{\omega}} e^{i\pi} - 1\right) - \frac{e^{i\nu t}}{\beta+i\omega-i\nu}\left(e^{(\beta-i\nu)\frac{\pi}{\omega}} e^{i\pi} - 1\right)\right\}$$

$$= -\frac{a}{2\nu}\, e^{-\beta t}\;\mathrm{Re}\left\{ \frac{1+e^{\beta\pi/\omega} e^{i\nu\pi/\omega}}{\beta+i\omega+i\nu}\, e^{-i\nu t} + \frac{1+e^{\beta\pi/\omega} e^{-i\nu\pi/\omega}}{\beta+i\omega-i\nu}\, e^{i\nu t}\right\}$$

$$= -\frac{a}{2\nu}\, e^{-\beta t}\;\mathrm{Re}\left\{ \frac{e^{-i\nu t} + e^{\pi\beta/\omega} e^{i\nu(\frac{\pi}{\omega}-t)}}{\sqrt{\beta^2+(\omega+\nu)^2}}\, e^{-i\tan^{-1}\left(\frac{\omega+\nu}{\beta}\right)} + \frac{e^{i\nu t} + e^{\pi\beta/\omega} e^{-i\nu(\frac{\pi}{\omega}-t)}}{\sqrt{\beta^2+(\omega-\nu)^2}}\, e^{-i\tan^{-1}\left(\frac{\omega-\nu}{\beta}\right)}\right\}$$

$$x(t) = -\frac{a}{2\nu} e^{-\beta t} \left\{ \frac{\cos\left(\nu t + \tan^{-1}\left(\frac{\omega+\nu}{\beta}\right)\right) + e^{\pi\beta/\omega} \cos\left(\nu\left(t - \frac{\pi}{\omega}\right) + \tan^{-1}\left(\frac{\omega+\nu}{\beta}\right)\right)}{\sqrt{\beta^2 + (\omega+\nu)^2}} \right.$$

$$\left. + \frac{\cos\left(\nu t + \tan^{-1}\left(\frac{\nu-\omega}{\beta}\right)\right) + e^{\pi\beta/\omega} \cos\left(\nu\left(t - \frac{\pi}{\omega}\right) + \tan^{-1}\left(\frac{\nu-\omega}{\beta}\right)\right)}{\sqrt{\beta^2 + (\omega-\nu)^2}} \right.$$

$\times\ H(t)$

response starts at $t = 0$