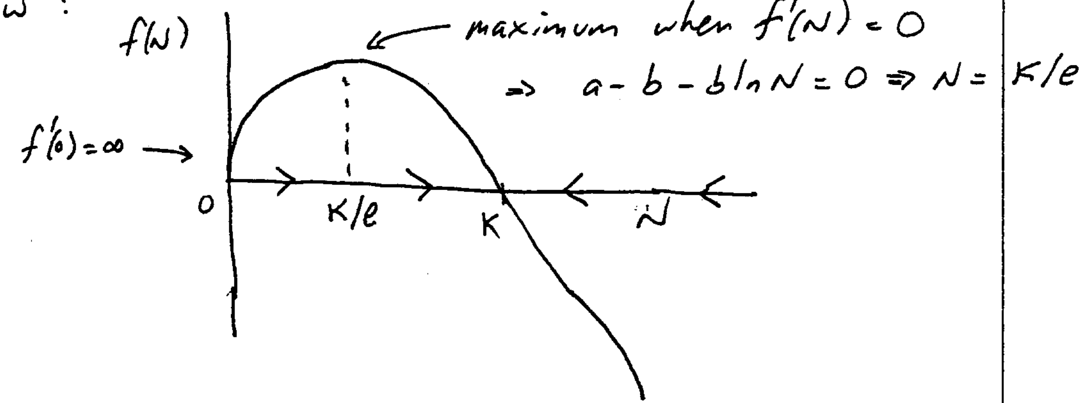
[1] $\dot{N} = aN - bN\ln N = f(N)$

$f(N) = 0 \Rightarrow N = 0, \quad \ln N = \frac{a}{b} \Rightarrow N = K \equiv \exp\left(\frac{a}{b}\right)$

Phase flow :



Analytic solution : set $y \equiv \ln N \Rightarrow$

$$\dot{y} = a - by$$

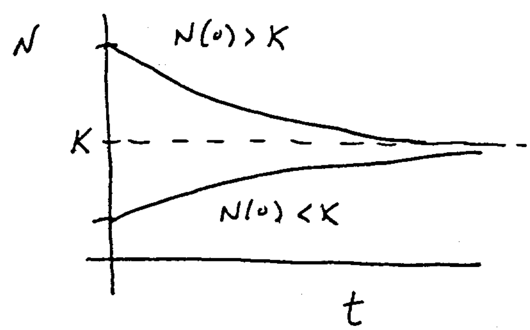
$$\frac{dy}{a-by} = -\frac{1}{b}\, d\ln(a-by) = dt$$

$$\ln\left[\frac{a-by(t)}{a-by(0)}\right] = -bt$$

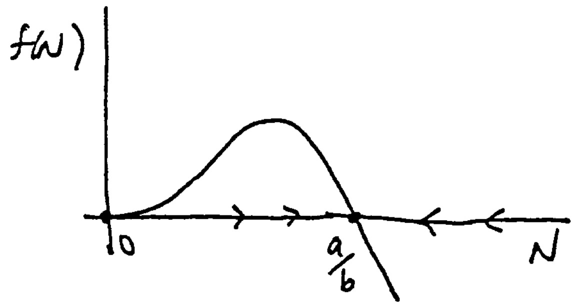
$$y(t) = \frac{a}{b} + \left(y(0) - \frac{a}{b}\right)e^{-bt}$$

$$\ln N = \ln K + \ln\left(\frac{N_0}{K}\right) e^{-bt}$$

$$N(t) = K \cdot \left(\frac{N(0)}{K}\right)^{e^{-bt}}$$



[2] $\dot{N} = aN^2 - bN^3 = N^2(a - bN)$. Fixed points at $N=0$, $N=\frac{a}{b}$.



$N=0$: unstable

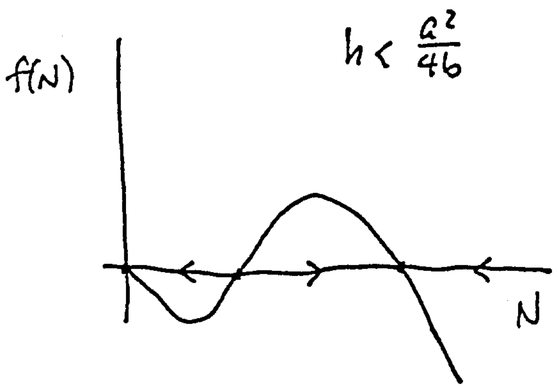
$N = \frac{a}{b}$: stable

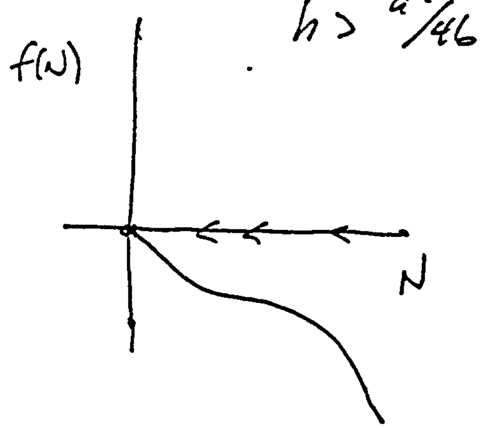
Harvested dynamics : $\dot{N} = -hN + aN^2 - bN^3$

Now $f(N) = -N(bN^2 - aN + h)$. Set $bN^2 - aN + h = 0$:

$$N_\pm = \frac{a \pm \sqrt{a^2 - 4bh}}{2b}$$

If $h > h_c = \frac{a^2}{4b}$, there are no real roots, and $f(N) < 0$, meaning $N^* = 0$ is the only fixed point. If $h < h_c$, we have three fixed points:





$N = 0$ (stable)

$N = \frac{a - \sqrt{a^2 - 4bh}}{2b}$ (unstable)

$N = \frac{a + \sqrt{a^2 - 4bh}}{2b}$ (stable).

$N = 0$ (stable)

For $h < \frac{a^2}{4b}$, we have

$$R = h N^*_{stable} = \frac{h}{2b} \cdot \left[a + \sqrt{a^2 - 4bh} \right]$$

$$\frac{\partial R}{\partial h} = \frac{1}{2b} \cdot \left\{ a + \sqrt{a^2 - 4bh} - \frac{1}{2} \frac{4bh}{\sqrt{a^2 - 4bh}} \right\}$$

$$\frac{\partial R}{\partial h} = 0 \Rightarrow a + \frac{3}{2}(a^2 - 4bh)^{1/2} - \frac{a^2}{2}(a^2 - 4bh)^{-1/2} = 0$$

$$\frac{3}{2}(a^2 - 4bh) + a(a^2 - 4bh)^{1/2} - \frac{a^2}{2} = 0$$

$$\Rightarrow (a^2 - 4bh)^{1/2} = \frac{-a \pm \sqrt{a^2 - 4 \cdot \frac{3}{2} \cdot \frac{a^2}{2}}}{3} = \cancel{-a}, \frac{a}{3}$$

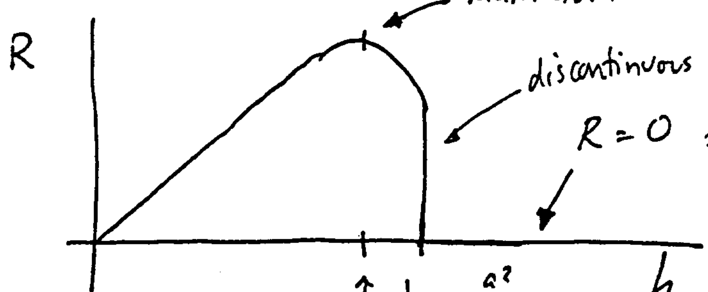
must take positive root!

Thus, $a^2 - 4bh = \frac{a^2}{9} \Rightarrow h^* = \frac{2a^2}{9b}$. Note $h^* < h_c$.

For $h = h^*$, $R = \frac{1}{2b} \cdot \frac{2a^2}{9b} \cdot \left[a + \frac{a}{3} \right] = \frac{4a^3}{27b^2}$.

For $h = h_c$, $R = \frac{1}{2b} \cdot \frac{a^2}{4b} \cdot a = \frac{a^3}{8b^2}$. Thus,

maximum harvest



discontinuous drop from $R(h_c^-) = \frac{a^3}{8b^2}$ to $R(h_c^+) = 0$.

$R = 0$ for $h > h_c$

$h_c = \frac{a^2}{9b}$

$h^* = \frac{8}{9} h_c = \frac{2a^2}{9b}$

[3] (a) $f(u) = a$ with $a \neq 0$ has no fixed points

$\dot{u} = a \Rightarrow u(t) = u(0) + at$

(b) $f(u) = 0$ has all $u \in \mathbb{R}$ as fixed points

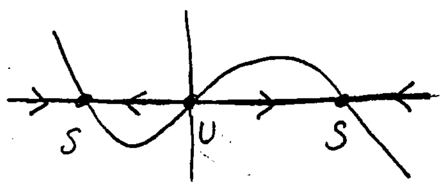
$\dot{u} = 0 \Rightarrow u(t) = u(0)$

(c) $f(u) = a \sin(\pi u)$ has all $u \in \mathbb{Z}$ as fixed points and no others ($a \neq 0$).

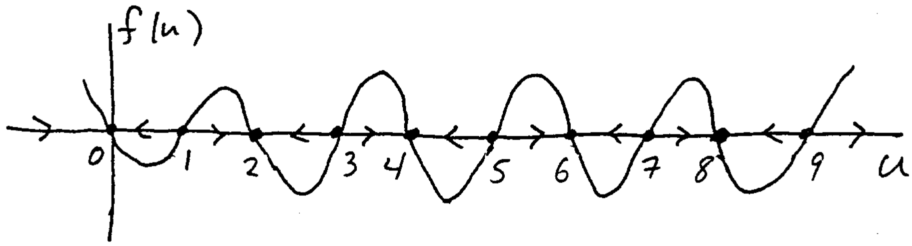
$$\dot{u} = a \sin(\pi u) \Rightarrow \frac{du}{\sin(\pi u)} = \frac{1}{\pi}\, d \ln \tan(\tfrac{1}{2}\pi u) = a\, dt$$

$$u(t) = \frac{2}{\pi} \tan^{-1}\left[\tan\left(\frac{\pi}{2} u(0)\right) e^{\pi a t} \right]$$

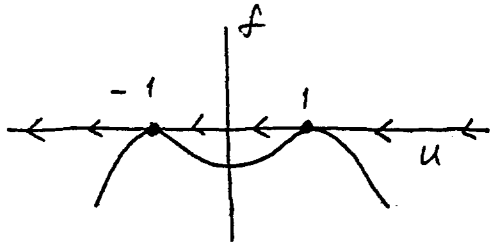
(d) It is impossible to have precisely three fixed points, all stable. Stable FPs alternate with unstable FPs. At best one could have two SFP and one UFP, e.g.



(e) $f(u) = u(u-1)(u-2)(u-3)(u-4)(u-5)(u-6)(u-7)(u-8)(u-9)$ has 10 FP at $u = 0, 1, \ldots, 9$:



(f) To have precisely two FP and $f(u) \le u \;\forall u$ is nongeneric. Example: $f(u) = -a\,(u^2-1)^2$ with $a>0$:



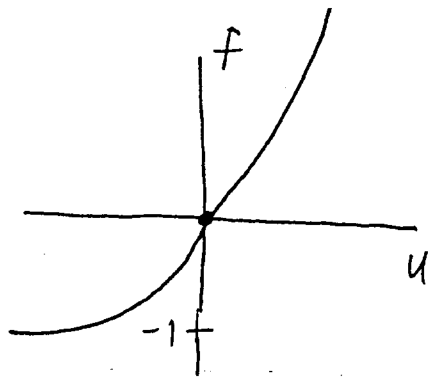
$u = \pm 1$ are both half-stable FP

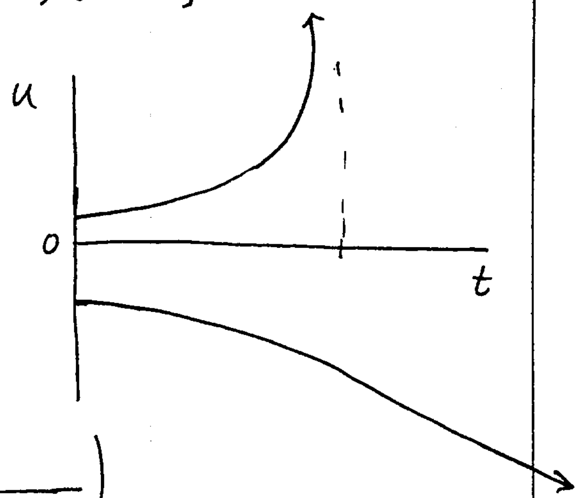
(g) In order that $u>0$ escape to $+\infty$ in finite time, $f(u)$ must grow faster than u for $u>0$. Similarly, in order that $u<0$ not escape to $-\infty$ in finite time, $f(u)$ cannot grow faster than u for $u<0$. An example satisfying both criteria is $f(u) = [e^u - 1]/\tau$ (note $f(0)=0$). Solution:

$\tau > 0$

$$\frac{du}{e^u - 1} = d\ln\left(1-e^{-u}\right) = dt/\tau$$

$$u(t) = -\ln\left[1 + \left(e^{-u(0)} - 1\right)e^{t/\tau}\right]$$

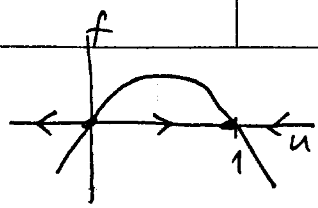




$$u(0) > 0 \;\Rightarrow\; t_\infty = \tau \ln\left(\frac{1}{1-e^{-u(0)}}\right)$$

$$u(0) < 0 \;\Rightarrow\; u(t) \sim -\frac{t}{\tau} - \ln\left(e^{-u(0)} - 1\right) + \dots \quad \text{as } t \to \infty$$

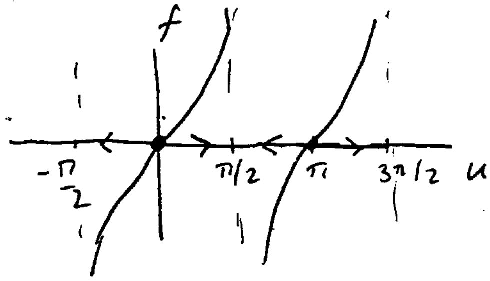
[4] (a) $\dot{u} = u(1-u)$



$u = 0$: unstable

$u = 1$: stable

(b) $\dot{u} = \tan u$

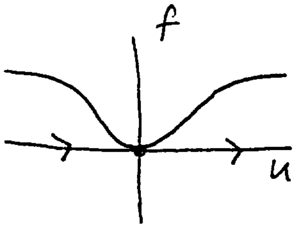


$u = \pi k \ (k \in \mathbb{Z})$: unstable

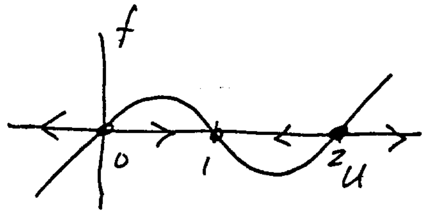
$u = \pi(k + \frac{1}{2})$: $f(u)$ undefined

(c) $\dot{u} = 1 - e^{-u^2}$

$u = 0$: half-stable



(d) $\dot{u} = u(1-u)(2-u)$

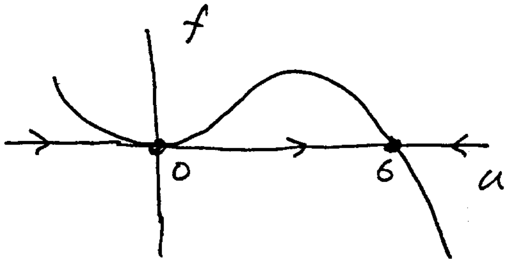


$u = 0$: unstable

$u = 1$: stable

$u = 2$: unstable

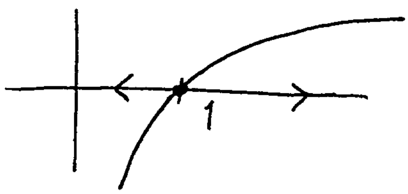
(e) $\dot{u} = u^2(6-u)$



$u = 0$: half-stable

$u = 6$: stable

(f) $\dot{u} = \ln u$

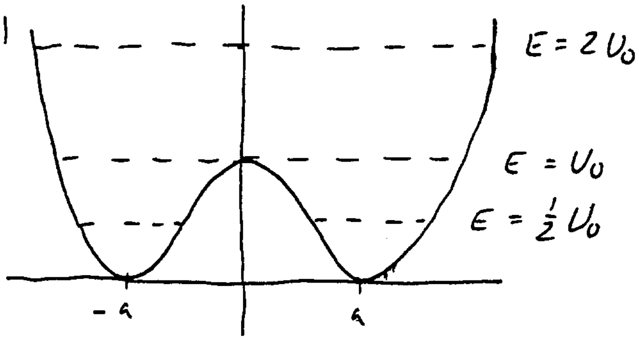


$u = 1$: unstable

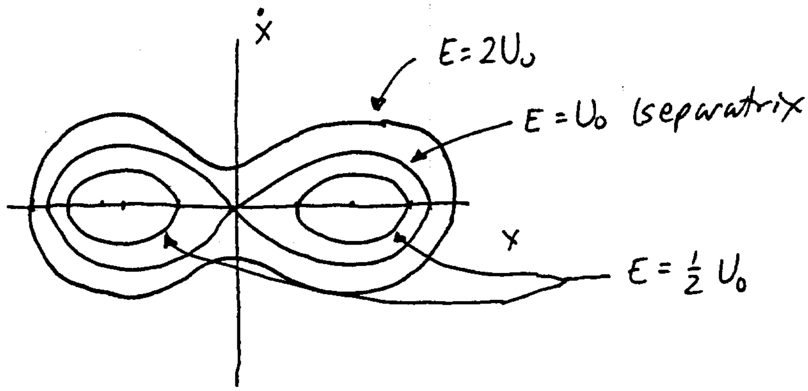
[5] $U(x) = U_0\left(\frac{x^2}{a^2} - 1\right)^2$

This function is non-negative and has minima at $x = \pm a$, where $U(\pm a) = 0$:

(a)



(b)



(c) Expand $U(x)$ in the vicinity of $x = a$:

$x \equiv a + \epsilon$ (ϵ small) $\Rightarrow$

$$U(a+\epsilon) = U(a) + U'(a)\,\epsilon + \tfrac{1}{2}U''(a)\,\epsilon^2 + \ldots$$

$$U(x) = U_0\left\{\frac{x^4}{a^4} - \frac{2x^2}{a^2} + 1\right\} \Rightarrow U(a) = 0,\ U'(a) = 0,\ U''(a) = \boxed{\frac{8U_0}{a^2}}$$

ϵ small

$$m\ddot{\epsilon} = -U''(a)\,\epsilon - \cancel{U'''(a)\,\epsilon^2} - \ldots$$

$$\Rightarrow \quad \omega = \sqrt{\frac{U''(a)}{m}} = \sqrt{\frac{8U_0}{ma^2}}$$

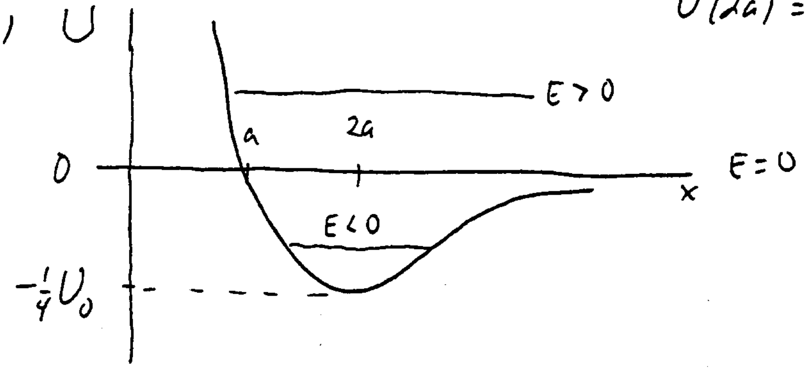
[6] $U(x) = U_0\left(\frac{a^2}{x^2} - \frac{a}{x}\right)$

$U(x) = 0 \Rightarrow x = a$; $U'(x) = U_0\left(-\frac{2a^2}{x^3} + \frac{a}{x^2}\right)$

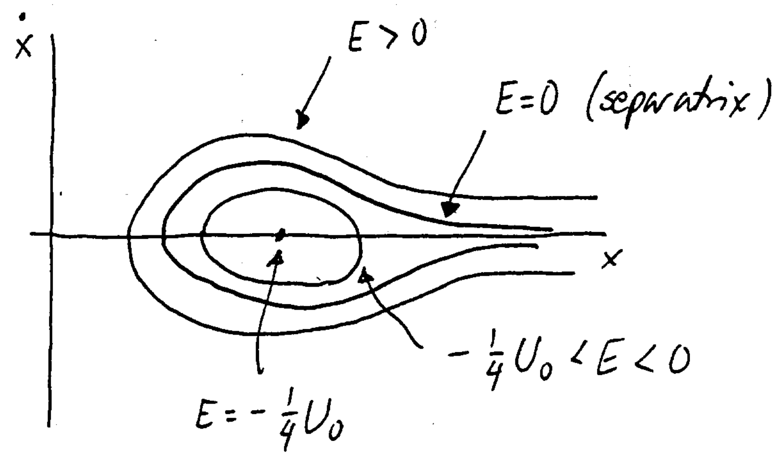
$U'(x) = 0 \Rightarrow x = 2a$

$U(2a) = U_{min} = -\frac{1}{4}U_0$

(a)



(b)



(c) We have

$$T(E) = \sqrt{2m}\int_{x_L(E)}^{x_R(E)} \frac{dx}{\sqrt{E - U(x)}}$$

where $x_{L,R}(E)$ are the turning points. Let's take

$$E = -\lambda U_0 \quad \text{with} \quad 0 < \lambda < \tfrac{1}{4}$$

First solve for turning points:

$$U(x) = U_0\left(\frac{a^2}{x^2} - \frac{a}{x}\right) = -\lambda U_0$$

Thus,

$$\lambda \left(\frac{x}{a}\right)^2 - \frac{x}{a} + 1 = 0$$

$$\frac{x}{a} = \frac{1 \pm \sqrt{1-4\lambda}}{2\lambda} \equiv S_\pm$$

The period is

$$T = \sqrt{2m} \int_{aS_-}^{aS_+} \frac{dx}{\sqrt{-\lambda U_0 + U_0 \frac{a}{x} - U_0 \frac{a^2}{x^2}}}$$

$$= \sqrt{\frac{2ma^2}{\lambda U_0}} \int_{S_-}^{S_+} \frac{s\, ds}{\sqrt{(S_+ - S)(S - S_-)}}$$

We can do the integral! Let

$$S \equiv \tfrac{1}{2}(S_+ + S_-) + \tfrac{1}{2}(S_+ - S_-)\, t$$

$$dS = \tfrac{1}{2}(S_+ - S_-)\, t$$

Find

$$T(\lambda) = \sqrt{\frac{2ma^2}{\lambda U_0}} \int_{-1}^{1} dt \; \frac{\frac{1}{2}(S_+ + S_-) + \frac{1}{2}(S_+ - S_-)\, t}{\sqrt{1+t^2}}$$

$$= \frac{\pi}{2} \sqrt{\frac{2ma^2}{U_0 \lambda^3}} \quad ; \qquad T(\lambda = \tfrac{1}{8}) = \frac{\pi}{32} \sqrt{\frac{ma^2}{U_0}}$$

Note that $U''(2a) = \frac{1}{8} \frac{U_0}{a^2}$ so near $x = 2a$ we have $\omega_0 = \left(U_0/8ma^2\right)^{1/2}$. Our expression gives for $\lambda = \frac{1}{4}$

$$T(\lambda = \tfrac{1}{4}) = 4\pi \sqrt{\frac{2ma^2}{U_0}} \Rightarrow \omega_0 = \frac{2\pi}{T} = \left(\frac{U_0}{8ma^2}\right)^{1/2} \checkmark$$