Physics 110A: Problem Set #1

Due Monday, October 1 by 12:30 pm

Reading: MT chapter 8 (review) ; Strogatz chs. 1 and 2; lecture note packets #1 and #2

[1] The growth of many types of cancerous tumors is adequately described by the Gompertz law,

$$\dot{N} = aN - bN \ln N \ ,$$

where N is the number of cells in the tumor ($N > 0$). Sketch the phase flow and identify and classify all fixed points. Solve for $N(t)$ given initial data $N(0)$.

[2] The bletch is a disgusting animal native to the Forest of Jkroo on the planet Barney. The bletch population obeys the dynamical equation

$$\frac{dN}{dt} = aN^2 - bN^3 \ ,$$

where N is the number of bletches, and a and b are positive constants. (Bletches reproduce asexually, but only when another bletch is watching. However, when there are three or more bletches around, they beat the `@!!*$&*` out of each other.)

(a) Sketch the phase flow for N (strange as the bletch is, you still can rule out $N < 0$). Identify all fixed points and classify them according to their stability.

(b) The bletch population is now *harvested* (they make nice shoes). To model this, we add an extra term to the dynamics:

$$\frac{dN}{dt} = -hN + aN^2 - bN^3 \ .$$

Show that the phase flow now depends crucially on h, in that there are two qualitatively different flows depending on whether $h < h_c(a,b)$ or $h > h_c(a,b)$. Find the critical value $h_c(a,b)$ and sketch the phase flows for $h < h_c$ and $h > h_c$.

(c) In equilibrium, the rate at which bletches are harvested is $R = hN^*$, where N^* is the equilibrium bletch population. Suppose we start with $h = 0$, in which case the equilibrium population is given by the value at the attractive fixed point you found in **(a)**. Now h is increased very slowly from zero. As h is increased, the equilibrium population changes. Sketch R *versus* h. What value of h achieves the biggest bletch harvest? What is the value of R_{max}? Show that R changes *discontinuously* at $h = h_c$.

[3] For each of the following cases, find an equation $\dot{u} = f(u)$ with the stated properties. If no such examples exist, explain why not. You may assume $f(u)$ is a smooth (i.e. infinitely differentiable) function in all cases.

(a) There are no fixed points.

(b) Every real number is a fixed point.

(c) Every integer is a fixed point, and there are no other fixed points.

(d) There are precisely three fixed points, and all of them are stable.

(e) There are precisely 10 fixed points.

(f) There are precisely two fixed points, and $f(u) \le 0 \; \forall u$.

(g) There is precisely one fixed point at $u = 0$. If $u(0) > 0$, then $u(t)$ flows to $u = \infty$ in a *finite* time. If $u(0) < 0$, then $u(t)$ flows to $u = -\infty$ only after an *infinite* time.

[4] Use linear stability analysis to classify the fixed points of the following systems. If the analysis fails because $f'(u^*) = 0$, use a graphical argument to determine the stability.

(a) $\dot{u} = u(1-u)$ **(d)** $\dot{u} = u(1-u)(2-u)$

(b) $\dot{u} = \tan u$ **(e)** $\dot{u} = u^2(6-u)$

(c) $\dot{u} = 1 - e^{-u^2}$ **(f)** $\dot{u} = \ln u$

[5] A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = U_0 \left(\frac{x^2}{a^2} - 1 \right)^2 .$$

(a) Sketch $U(x)$, identifying any minima and/or maxima.

(b) Sketch the phase curves for a particle with energy $E_1 = \frac{1}{2} U_0$, $E_2 = U_0$, and $E_3 = 2U_0$.

(c) Compute the frequency of oscillation when x is close to a.

[6] A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = U_0 \left\{ \frac{a^2}{x^2} - \frac{a}{x} \right\} .$$

(a) Sketch $U(x)$, identifying any minima and/or maxima.

(b) Sketch a set of (at least four, say) phase curves at various representative energies. Make sure to identify any fixed points and separatrices, and find their energies.

(c) For the phase curve with energy $E = -\frac{1}{8} U_0$, derive an integral expression for the period T. Your expression should take the form $T = \tau \cdot \mathcal{I}$, where $\tau = \sqrt{ma^2/U_0}$ is the natural time scale for this system (the only quantity with dimensions of time which can be formed from a, U_0, and m), and $\mathcal{I}$ is a dimensionless integral.