PHYSICS 110A
SOLUTION SET 5

1) $u(r) = U_0 \ln(r/a)$

(a) $E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{l^2}{mr^2} + U_0 \ln(r/a)$

$\therefore U_{eff} = \frac{l^2}{2mr^2} + U_0 \ln(r/a)$

(b) $\frac{du}{dr} = 0 \Rightarrow \quad \frac{-2l^2}{2mr_0^3} + \frac{U_0 a}{r_0} = 0$

$r_0 = \sqrt{\frac{l^2 m}{U_0 a}}$

$2\pi r_0 = \dot{\phi} \tau_0 \qquad ; \dot{\phi} = \frac{l}{mr_0^2}$

$\tau_0 = \frac{2\pi r_0}{\dot{\phi}} = \frac{2\pi m r_0^3}{l}$

$= \frac{2\pi l^2 m^2}{U_0 a} \sqrt{\frac{m}{U_0 a}}$

(c) $r(t) = r_0 + \eta(t)$

$$L = T - U = \frac{1}{2} m \dot{\eta}^2 + \frac{l^2}{2m} \left(\frac{1}{r_0^2} - \frac{2}{r_0^3}\eta + \frac{3}{r_0^4}\eta^2 + \mathcal{O}(\eta^3) \right)$$
$$- U_0 \left(\ln \frac{x_0}{a} + \frac{a}{r_0}\eta - \frac{a}{2r_0^2}\eta^2 + \mathcal{O}(\eta^3) \right)$$

$(\eta): \; m\ddot{\eta} + \eta\left[\frac{aU_0}{r_0^2} + \frac{3l^2}{mr_0^4}\right] = 0$

NOTE THAT SINCE $\left.\frac{dU}{dr}\right|_{r=r_0} = 0$, THE TERMS LINEAR TO η IN MY LAGRANGIAN GO AWAY.

FOR η SMALL, WE GET SIMPLE HARMONIC MOTION WITH

$$\omega^2 = \frac{aU_0}{mr_0^2} + \frac{3l^2}{m^2 r_0^4}$$

SO, $\eta(t) = A\cos \omega t + \delta$, FOR CONVENIENCE, SAY $\delta = 0$.

(d) $t = \phi/\dot{\phi}$, $\therefore \eta(\phi) = A\cos\left(\frac{\omega\phi}{\dot{\phi}}\right)$, WITH $\dot{\phi}$ AS DISCUSSED ABOVE.

TO DISCUSS A "CLOSED ORBIT", CHECK IF $\eta(\phi=0) = \eta(\phi = 2\pi)$, OR EQUIVALENTLY, IF $\eta(t=0) = \eta(t=\tau_0)$

$$\eta(0) = A$$

$$\eta(t=\tau_0) = A\cos\left[\underbrace{\left(\sqrt{\frac{aU_0}{mr_0^2} + \frac{3l^2}{m^2 r_0^4}}\right)\frac{2\pi l^2 m^2}{U_0 a}\sqrt{\frac{m}{U_0 a}}}_{=2\pi n}\right]$$

SO, I NEED

OR $$n \stackrel{?}{=} \sqrt{\frac{aU_0}{mr_0^2} + \frac{3l^2}{m^2 r_0^4}}\;\frac{l^2 m^2}{U_0 a}\sqrt{\frac{m}{U_0 a}} \quad \text{WITH } n \in \mathbb{Z}$$

I'LL GIVE YOU A HINT — IT DOESN'T.

2) $r(\theta) = \dfrac{r_0}{1+\varepsilon\cos\theta}$

$$r_0 \equiv \frac{l^2}{\mu G M m}$$

$$\varepsilon \equiv \sqrt{1+\frac{2l^2E}{\mu(GMm)^2}} = \sqrt{1+\frac{2l^4}{\mu^2(GMm)^2}-\frac{2l^2 GMm}{\mu(GMm)^2 r_0}}$$

$$= \sqrt{\frac{2l^4}{\mu^2G^2M^2m^2}-1}$$

SO, IF $\varepsilon=0$ ~~AT~~ INITIAL CONDITION, WE GET A CIRCULAR ORBIT.

NOW, AS α INCREACES, M GETS SMALLER, AND WE GET

ELIPTICAL ORBITS, UNTIL $\varepsilon=1$;

OR $\sqrt{\dfrac{2l^4}{\mu^2G^2M^2m^2\alpha^2}-1}=1$. AT THIS POINT, WE GET

A PARABOLIC ORBIT. FOR $\varepsilon>1$, WE GET HYPERBOLAE.

3) $r(\phi) = C\phi^2$

WE KNOW THAT

$$\frac{d^2}{d\phi^2}\left(\frac{1}{C\phi^2}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)$$

THUS, $$F(r) = -\frac{l^2}{\mu r^2}\left[\frac{1}{r} + \frac{6}{C\phi^4}\right]$$

$$= \underline{\frac{-l^2}{\mu r^2}\left[\frac{1}{r} + \frac{6C}{r^2}\right]}$$

4) SINCE WE ARE REQUIRING OUR SPACE STATION TO BE "GEOSYNCHRONUS," THIS TELLS US IMMEDIATELY THAT IT MUST BE OVER THE EQUATOR. OTHERWISE THE SATELITE WOULD OSCILATE BETWEEN THE SOUTHERN AND NORTHERN HEMISPHERES.

WE ALSO REQURE $F_G = F_c$, OR $m_s \omega^2 r = \frac{+GM_E m_s}{r^2}$,

OR $r = \left(\frac{GM}{\omega^2}\right)^{1/3}$, WITH $\omega = 2\pi/T$

5) $r = \dfrac{r_0}{1+\varepsilon\cos\phi}$

NOW, WE KNOW

$$r_{MAX} = \frac{r_0}{1-\varepsilon}$$

$$r_{MIN} = \frac{r_0}{1+\varepsilon},$$

WE ALSO KNOW $\ell \equiv \mu r^2\dot{\phi} = \text{CONST}$

$$\therefore \dot{\phi} = \frac{\ell}{\mu r^2}, \quad \text{AND} \quad \dot{\phi}_{MAX} = \frac{\ell}{\mu r_{min}^2} \quad ; \quad \dot{\phi}_{MIN} = \frac{\ell}{\mu r_{max}^2}$$

THUS, $\dfrac{\dot{\phi}_{max}}{\dot{\phi}_{MIN}} \equiv \lambda = \dfrac{r_{max}^2}{r_{MIN}^2} = \dfrac{(1+\varepsilon)^2}{(1-\varepsilon)^2}$

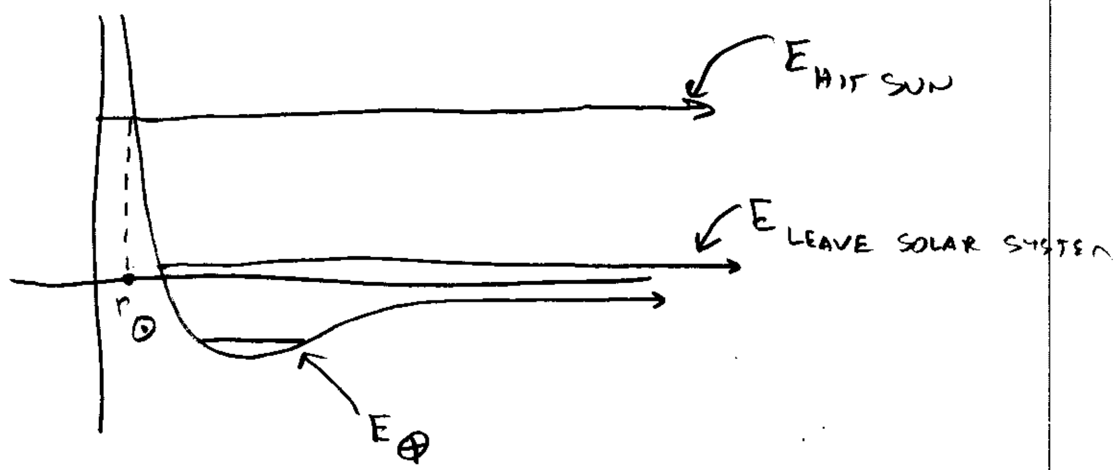
$$\sqrt{\lambda}(1-\varepsilon) = (1+\varepsilon)$$

$$\varepsilon(1+\sqrt{\lambda}) = \sqrt{\lambda} - 1$$

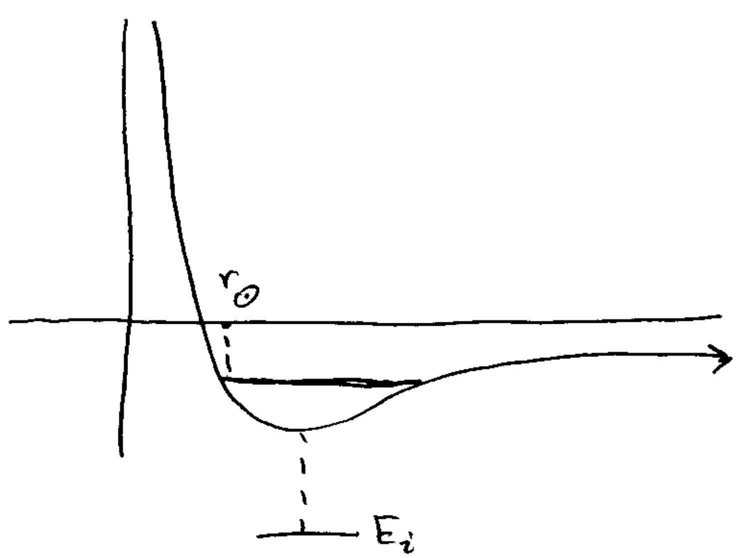
$$\boxed{\varepsilon = \frac{\sqrt{\lambda}-1}{\sqrt{\lambda}+1}}$$

(6) LET US CONSIDER A QUALATATIVE ANALYSIS OF THIS QUESTION.

SPECIFICALLY, LET US LOOK AT $\mathcal{U}_{eff}(r)$



IF WE THRUST IN THE DIRECTION OF THE EARTH'S ORBIT, IT TAKES A NEAR TRIVIAL AMOUNT OF ENERGY TO ENTER A HYPERBOLIC ORBIT, WHEREAS HITTING THE SUN REQUIRES A LOT OF ENERGY, BECAUSE OF THE "CENTRIFUGAL BARRIER."



THRUSTING THE OPPOSITE DIRECTION IS HARDER — FIRST YOU HAVE TO OVERCOME YOUR INERTIA, THEN ENTER AN ELLIPTICAL ORBIT AROUND THE SUN.
IT IS EASIER TO LEAVE THE SOLAR SYSTEM.

7) (a) $\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{-\mu r^2}{l^2} f(r)$

$$= \frac{d^2}{d\phi^2}\left(\frac{\cos(\phi-\phi_0)}{a}\right) + \frac{1}{r} = \frac{-\mu r^2}{l^2} f(r)$$

$$= \frac{-\cos(\phi-\phi_0)}{a} + \frac{1}{r} = \frac{-\mu r^2}{l^2} f(r)$$

$$= 0$$

THIS MAKES PERFECT SENSE, AS $r = \frac{a}{\cos(\phi-\phi_0)}$ IS THE EQUATION OF A STRAIGHT LINE.

(b) $r(\phi) = \frac{2b}{\phi^2}$

$$f(r) = \frac{-l^2}{\mu r^2}\left[\frac{1}{r} + \frac{1}{2b}\right]$$

SINCE $f(r) < 0$ FOR ANY CHOICE OF r $(r > 0)$, THIS IS AN ATTRACTIVE FORCE