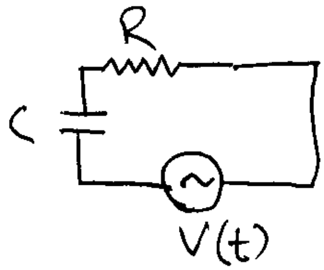
PHYSICS 110A PROBLEM SET #2

1(a)



$$RI + \frac{Q}{C} = V(t)$$

$$\dot{Q} + \frac{Q}{RC} = \frac{V(t)}{R}$$

(b) $\dot{Q} + \frac{Q}{RC} = 0$

$$\frac{dQ}{Q} = \frac{-dt}{RC}$$

$$Q = C_1 e^{-\frac{t}{RC}}$$, AND SINCE $Q(t=0) = Q_0$,

$$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

(c) $\dot{Q} + \frac{Q}{RC} = \frac{V(t)}{R}$

SO, LET'S FIND OUR GREEN'S FUNCTION.

FIRST, FOURIER TRANSFORM BOTH SIDES:

$$-i\omega \hat{Q}(\omega) + \frac{1}{RC}\hat{Q}(\omega) = \frac{\hat{V}(\omega)}{R}$$

$$\hat{Q}(\omega) = \frac{\hat{V}(\omega)}{R} \frac{1}{\frac{1}{RC} - i\omega}$$

$$\hat{Q}(\omega) = C\,\hat{V}(\omega)\,\frac{1}{1 - i\omega RC}$$

NOW, INVERT THE TRANSFORM:

$$Q_p(t) = C\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{\hat{V}(\omega)\,e^{-i\omega t}}{1 - i\omega RC}$$

$$= C\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}dt'\,V(t')\,\frac{e^{-i\omega(t-t')}}{1 - i\omega RC}$$

$$= C\int_{-\infty}^{\infty}dt'\,V(t')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega(t-t')}}{1 - i\omega RC}$$

THE SECOND INTEGRAL IS OUR GREEN'S FUNCTION

SO, WE QUOTE THE GIVEN INTEGRAL ON THE

HOMEWORK ~~(FOR INTERESTED PARTIES, I HAVE ATTACHED THE ACTUAL INTEGRAL SOLUTION AT THE END OF THE HOMEWORK SOLUTION)~~

$$Q_p(t) = C\int_{-\infty}^{\infty}dt'\,V(t')\,\frac{1}{RC}\,\Theta(t-t')\,e^{-(t-t')/RC} \qquad (1)$$

* NOW, WE PLUG IN $V(t) = V_0\,\Theta(t)$

$$= \frac{1}{R}\int_0^t dt'\,V_0\,e^{-(t-t')/RC}$$

$$= \frac{V_0}{R}\cdot(\cancel{-}RC)\,e^{-(t-t')/RC}\Big|_0^t$$

$$\cancel{= V_0 C\left[e^{-t/RC} - 1\right]}$$

$$= \frac{V_0}{R}\cdot RC\left[1 - e^{-t/RC}\right]$$

SO, $Q(t) = Q_p(t) + Q_H(t)$

$$= Q_0 e^{-t/RC} + V_0 C\left[1 - e^{-t/RC}\right]$$

AND $I = \dot{Q} = \dfrac{-Q_0}{RC} e^{-t/RC} + \dfrac{V_0}{R} e^{-t/RC}$

(d) WE CAN QUOTE (1) FROM PART C. THEN,

$$Q_p(t) = C\int_{-\infty}^{\infty} dt'\, V(t') \frac{1}{RC} \Theta(t-t') e^{-(t-t')/RC}$$

$$= \frac{1}{R}\int_{-\infty}^{\infty} V_0\, \Theta(t')\Theta(t-t') \frac{1}{2i}\left[e^{i\Omega t'} - e^{-i\Omega t'}\right] e^{-(t-t')/RC} dt'$$

$$= \frac{V_0}{2iR} e^{-t/RC}\int_0^{t} dt' \left[e^{\left(\frac{1}{RC}+i\Omega\right)t'} - e^{\left(\frac{1}{RC}-i\Omega\right)t'}\right.$$

$$= \frac{V_0}{2iR} e^{-t/RC}\left[\frac{1}{\frac{1}{RC}+i\Omega}\left[e^{\left(\frac{1}{RC}+i\Omega\right)t} - 1\right] - \frac{1}{\frac{1}{RC}-i\Omega}\left[e^{\left(\frac{1}{RC}-i\Omega\right)t} - 1\right]\right]$$

$$= \frac{V_0}{2iR} e^{-t/RC}\left[\frac{\frac{1}{RC}-i\Omega}{\left(\frac{1}{RC}\right)^2+\Omega^2}\left[e^{\left(\frac{1}{RC}+i\Omega\right)t} - 1\right] - \frac{\frac{1}{RC}+i\Omega}{\left(\frac{1}{RC}\right)^2+\Omega^2}\left[e^{\left(\frac{1}{RC}-i\Omega\right)t} - 1\right]\right]$$

$$\cancel{= \frac{V_0 e^{-t/RC}}{2iR}\left[\frac{2i\Omega}{\left(\frac{1}{RC}\right)^2+\Omega^2} + \left[\left(\frac{1}{RC}\right)^2+\Omega^2\right]^{-1}\right]}$$

$$= \frac{V_0 e^{-t/RC}}{2iR\left(\frac{1}{R^2C^2}+\Omega^2\right)}\left[2i\Omega + e^{\frac{t}{RC}}\left[\frac{1}{RC}\left(e^{i\Omega t}-e^{-i\Omega t}\right) - i\Omega\left[e^{i\Omega t}+e^{-i\Omega t}\right]\right]\right]$$

$$= \frac{V_0 e^{-t/RC}}{2iR\left(\frac{1}{R^2C^2}+\Omega^2\right)}\left[2i\Omega + e^{t/RC}\cdot\frac{2i}{RC}\sin\Omega t - 2i\Omega\cos\Omega t\right]$$

$$\cancel{= \frac{V_0 e^{-t/RC}}{R\left(\frac{1}{R^2C^2}+\Omega^2\right)}\left[\Omega\left(1-\cos\Omega t\right)\right]}$$

$$= \frac{V_0 e^{-t/RC}}{R\left(\frac{1}{R^2C^2}+\Omega^2\right)}\left[\Omega\left(1 - e^{t/RC}\cos\Omega t\right) + \frac{e^{t/RC}}{RC}\sin\Omega t\right]$$

2) ALL WE ARE INTERESTED IN IS THE RESPONSE FUNCTION, WHICH WE CAN GET BY SOLVING

$$x_p(t) = \int_{-\infty}^{\infty} f(t')\, G(t-t')\, dt'$$

$$= \int_{-\infty}^{\infty} dt'\, f_0\, e^{-\gamma t'}\cos(\Omega t')\, e^{-\beta(t-t')}\sin(\nu(t-t'))\,\Theta(t')\,\Theta(t-t')$$

$$= f_0\int_0^t dt'\, e^{-\gamma t'} e^{-\beta(t-t')}\left[\frac{1}{2}\left(e^{i\Omega t'}+e^{-i\Omega t'}\right)\right]\left[\frac{1}{2i}\left(e^{i\nu(t-t')}-e^{-i\nu(t-t')}\right)\right]$$

$$= \frac{f_0 e^{-\beta t}}{4i}\left[e^{-i\nu t}\int_0^t dt\left\{e^{t'[\beta-\gamma+i(\Omega+\nu)]}+e^{t'[\beta-\gamma+i(\nu-\Omega)]}\right]\right.$$
$$\left. + e^{i\nu t}\int_0^t dt'\left[e^{t'[\beta-\gamma+i(\Omega-\nu)]}+e^{t'[\beta-\gamma-i(\nu+\Omega)]}\right]\right]$$

...

WHICH IS AN EASY INTEGRAL TO DO.

$$x_p(t) = \frac{f_0 e^{-\beta t}}{4i}\left[\left.\frac{e^{-i\nu t}\, e^{t'[\beta-\gamma+i(\Omega+\nu)]}}{\beta-\gamma+i(\Omega+\nu)}\right|_{t'=0}^{t} + \left.\frac{e^{-i\nu t}\, e^{t'[\beta-\gamma+i(\nu-\Omega)]}}{\beta-\gamma+i(\nu-\Omega)}\right|_{t'=0}^{t} \right.$$

$$\left. + \left.\frac{e^{i\nu t}\, e^{t'[\beta-\gamma+i(\Omega-\nu)]}}{\beta-\gamma+i(\Omega-\nu)}\right|_{t'=0}^{t} + \left.\frac{e^{i\nu t}\, e^{t'[\beta-\gamma-i(\Omega+\nu)]}}{\beta-\gamma-i(\Omega+\nu)}\right|_{t'=0}^{t} \right]$$

NOW, IF YOU GOT TO HERE, YOU DID FINE. FROM HERE TO THE FINAL SOLUTION IS LOTS OF HIDEOUS ALGEBRA. SO, I WILL SIMPLY QUOTE THE SOLUTION FOR INTERESTED PARTIES:

$$x_p(t) = f_0 e^{-\beta t} \frac{\left\{ -\nu\left[(\beta-\gamma)^2+\nu^2-\Omega^2\right]\cos\nu t - (\beta-\gamma)\left[(\beta-\gamma)^2+\nu^2+\Omega^2\right]\sin\nu t + \nu e^{t(\beta-\nu)}\left[\left[(\beta-\gamma)^2+\nu^2-\Omega^2\right]\cos\Omega t + 2(\beta-\gamma)\Omega\sin\Omega t\right]\right\}}{\left[\left[(\beta-\gamma)^2+\nu^2\right]^2 + 2\left[(\beta-\gamma)^2-\nu^2\right]\Omega^2+\Omega^4\right]}$$

3) $\ddot{x} + \omega^2(t)x = 0$ WHERE $\omega(t) = \begin{cases} \omega_0 & t \in [2n\tau, (2n+1)\tau];\ n \in \mathbb{Z} \\ 0 & t \in [(2n+1)\tau, 2(n+1)\tau];\ n \in \mathbb{Z} \end{cases}$

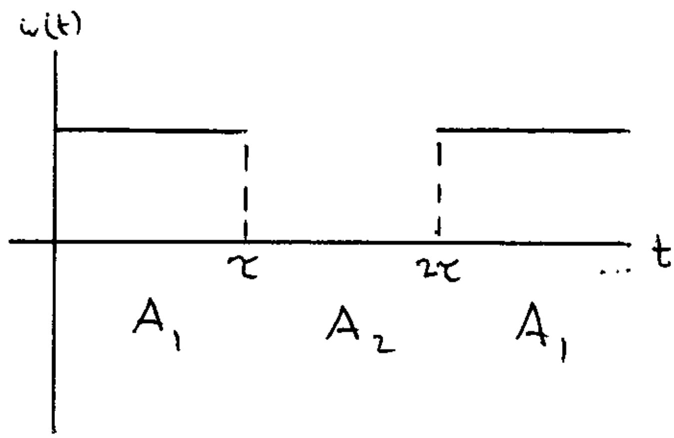
SO, WE CAN WRITE THIS AS $\dot{\vec{\varphi}} = A\vec{\varphi}$

$$\begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

OR, $\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \cancel{\begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}}\, e^{\int_0^t dt' A(t')} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix}$

~~SO, LET'S CONSIDER THE FIRST PART OF~~

LET'S LOOK AT $\omega(t)$ AS A GRAPH:



APPLICABLE MATRIX:

SO, $A(t)$ DEPENDS ON WHAT PART OF THE PERIOD WE'RE IN.

LET'S TAKE $\vec{\varphi}(\tau)$ BASED ON $\vec{\varphi}(0)$

$$\vec{\varphi}(\tau) = e^{A_1\tau}\vec{\varphi}(0)$$

AND $\vec{\varphi}(2\tau) = e^{A_2\tau}\vec{\varphi}(\tau) = e^{A_2\tau}e^{A_1\tau}\vec{\varphi}(0)$

$\therefore$ MY PROPAGATION MATRIX FOR ONE PERIOD IS $e^{A_2\tau}e^{A_1\tau}$

WITH $A_1 = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}$

$$A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

So, $e^{A_1\tau} = \begin{pmatrix} \cos\omega_0\tau & \frac{1}{\omega_0}\sin\omega_0\tau \\ -\omega_0\sin\omega_0\tau & \cos\omega_0\tau \end{pmatrix}$

AND $e^{A_2\tau} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$

$$\therefore\ e^{A_2\tau}e^{A_1\tau} = \mathcal{M} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\omega_0\tau & \frac{1}{\omega_0}\sin\omega_0\tau \\ -\omega_0\sin\omega_0\tau & \cos\omega_0\tau \end{pmatrix}$$

$$= \begin{pmatrix} \cos\omega_0\tau - \omega_0\tau\sin\omega_0\tau & \frac{1}{\omega_0}\sin\omega_0\tau + \tau\cos\omega_0\tau \\ -\omega_0\sin\omega_0\tau & \cos\omega_0\tau \end{pmatrix}$$

FOR UNBOUNDED MOTION, I REQUIRE

$$\begin{pmatrix} \varphi_\infty \\ \dot\varphi_\infty \end{pmatrix} = \lim_{n\to\infty} \mathcal{M}^n \vec{\varphi}_0 \ \text{~~[illegible]~~ AND } \varphi_\infty = \infty$$

–OR–, I CAN REQUIRE $|\mathrm{Tr}(\mathcal{M})| > 2$ THIS WOULD REQUIRE ~~Areal~~ $\lambda \in \mathrm{Re}$, AND AS SUCH, A GROWING SOLUTION.

So, $\mathrm{Tr}\,\mathcal{M} = \cos\theta - \theta\sin\theta + \cos\theta$

$\cancel{\text{So } 2\cos\theta > \theta\sin\theta}$

$\cancel{\boxed{\frac{2}{\theta} > \tan\theta}}$

$$\therefore\ |2\cos\theta - \theta\sin\theta| > 2$$

4) $\ddot{x} + \omega_0^2 x = -\omega_0^2 x^2 = -\varepsilon\omega_0^2 x^2$

TAKE $t = \tau/\Omega$

THEN $\ddot{x} = \Omega^2 x''$, WHERE $x'' = \frac{d^2x}{d\tau^2}$

~~ABD~~ NOW, EXPAND $x = x_0 + \varepsilon x_1 + \cdots + \varepsilon^n x_n + \cdots$

$$\Omega^2 = \Omega_0^2 + \varepsilon\Omega_1^2 + \cdots + \varepsilon^n \Omega_n^2$$

THUS, $\Omega^2 x'' + \omega_0^2 x = -\varepsilon\omega_0^2 x^2$

CAN BE WRITTEN

$$\left(\sum_m \varepsilon^m \Omega_m^2\right)\left(\sum_n \varepsilon^n x_n''\right) + \omega_0^2\left(\sum_n \varepsilon^n x_n\right) = -\varepsilon\omega_0^2\left(\sum_n \varepsilon^n x_n\right)^2$$

LET'S TAKE THE ~~ε^0~~ ε^0 TERMS:

$$\Omega_0^2 x_0'' + \omega_0^2 x_0 = 0$$

$$x_0 = Ae^{i\frac{\omega_0}{\Omega_0}\tau} + A^* e^{-i\frac{\omega_0}{\Omega_0}\tau} \quad \text{AND, TAKE } \Omega_0 = \omega_0$$

NOW, TAKE ε^1 TERMS:

$$\Omega_1^2 x_0'' + \Omega_0^2 x_1'' + \omega_0^2 x_1 = -\omega_0^2 x_0^2$$

NOW, CHOOSE Ω_1^2 S.T $\Omega_1^2 x_0'' = -\omega_0^2 x_0^2$

$$\Omega_1^2\left[+Ae^{i\tau} + A^* e^{-i\tau}\right] = +\omega_0^2\left[Ae^{i\tau} - Ae^{-i\tau}\right]$$

THUS, $\Omega_1^2 = \omega_0^2 x_0$

AND $x_1 = Be^{i\tau} + B^* e^{-i\tau}$

NOW, WE WILL ONLY LOOK AT THE ε^2 TERMS:

$$\varepsilon^2: \quad \Omega_0^2 x_2'' + \Omega_1^2 x_1'' + \Omega_2^2 x_0'' + \omega_0^2 x_2 = -2\omega_0^2 x_1 x_0$$

NOW I CANNOT PICK Ω_2^2 NICELY FOR THIS TO ALL WORK OUT.

~~CONTOUR INTEGRAL~~

$$\cancel{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{1-iRC\omega} \Rightarrow \oint_C \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{1-iRC\omega} =}$$