Physics 110A: Problem Set #2

`Reading: MT chapters 3 and 4; class notes pp. 33-60.`

[1] An electrical circuit consists of a resistor R and a capacitor C connected in series to an emf $V(t)$.

(a) Write down a differential equation for the charge $Q(t)$ on one of the capacitor plates.

(b) Solve the homogeneous equation for $Q(t)$ (that is, the solve for $Q(t)$ when $V(t) = 0$).

(c) Solve for the current flowing in the circuit when $V(t) = V_0\,\Theta(t)$.

(d) Solve for the current flowing in the circuit when $V(t) = V_0\,\Theta(t)\,\sin(\Omega t)$.

For parts **(c)** and **(d)**, you should use the Green's function formalism in the time domain. The following integral may prove useful:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\frac{\exp(-i\omega s)}{1 - i\omega\tau} = \frac{1}{\tau}\,\Theta(s)\,\exp(-s/\tau)\ .$$

[2] A forced damped harmonic oscillator obeys the equation of motion

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)\ .$$

Compute the response $x(t)$ to the forcing function

$$f(t) = f_0\,e^{-\gamma t}\,\cos(\Omega t)\,\Theta(t)\ .$$

[3] Consider the parametric oscillator

$$\ddot{x} + \omega^2(t)\,x = 0$$

where

$$\omega(t) = \begin{cases} \omega_0 & t \in [2n\tau, (2n+1)\tau] \\ 0\ , & t \in [(2n+1)\tau, 2(n+1)\tau] \end{cases}\ .$$

Compute the time evolution matrix for this problem for the interval $[2n\tau, 2(n+1)\tau]$ and determine for which values of $\theta \equiv \omega_0\tau$ is the motion unbounded.

[4] [EXTRA CREDIT] A nonlinear oscillator obeys the equation of motion

$$\ddot{x} + \omega_0^2 x = -\omega_0^2 x^2 \ .$$

Compute the frequency shift for small oscillations up to second order in the amplitude, and write down the motion of the system.

Remarks: Use Lindstedt's method for the equation $\ddot{x} + \omega_0^2 x = -\epsilon\, \omega_0^2 x^2$, with $\epsilon = 1$ at the end of the calculation. That is, write $t = \tau/\Omega$ where τ is dimensionless and Ω is the oscillation frequency, and expand both $x(\tau)$ as well as Ω^2 as a power series in ϵ:

$$x(\tau) = x_0(\tau) + \epsilon\, x_1(\tau) + \epsilon^2\, x_2(\tau) + \ldots$$
$$\Omega^2 = \Omega_0^2 + \epsilon\, \Omega_1^2 + \epsilon^2\, \Omega_2^2 + \ldots$$

to obtain the equation

$$\left(\sum_{m=0}^{\infty} \epsilon^m\, \Omega_m^2\right)\left(\sum_{n=0}^{\infty} \epsilon^n\, \frac{d^2 x_n}{d\tau^2}\right) + \omega_0^2 \left(\sum_{n=0}^{\infty} \epsilon^n\, x_n\right) = -\epsilon\, \omega_0^2 \left(\sum_{n=0}^{\infty} \epsilon^n\, x_n\right)^2$$

and choose your coefficients Ω_n^2 in such a way as to remove the resonant forcing terms at each order of perturbation theory. In this particular case of a quadratic nonlinearity, you should find *no* resonant forcing to order ϵ, but you will find such a term at order ϵ^2.