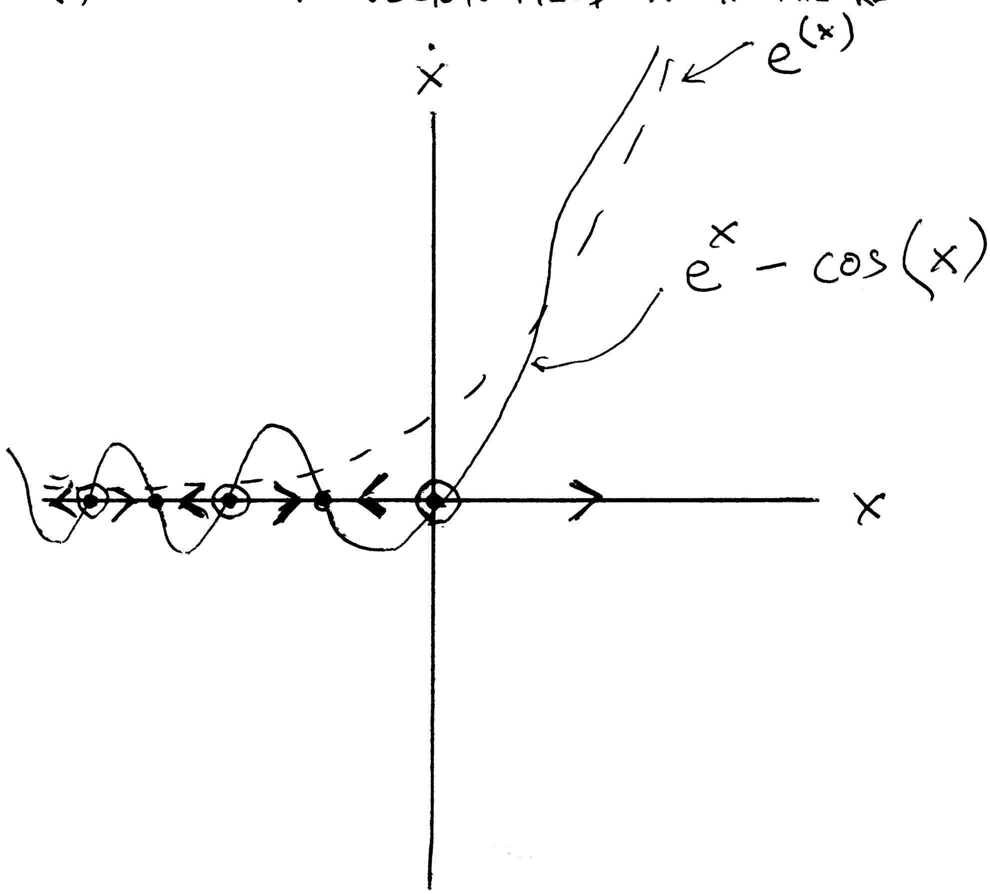
PHYS 110A HOMEWORK I

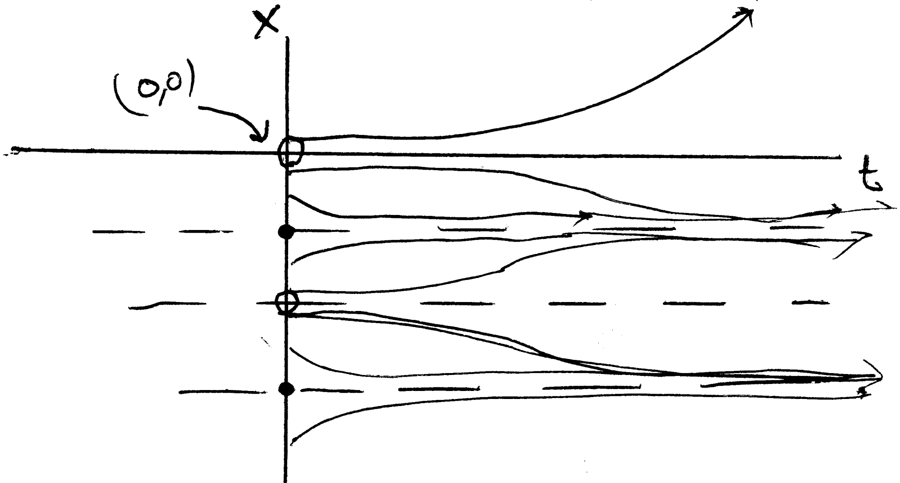
1. CONSIDER $\dot{x} = e^{x} - \cos x$

(a) SKETCH THE VECTOR FIELD $\dot{x}$ ON THE REAL LINE:

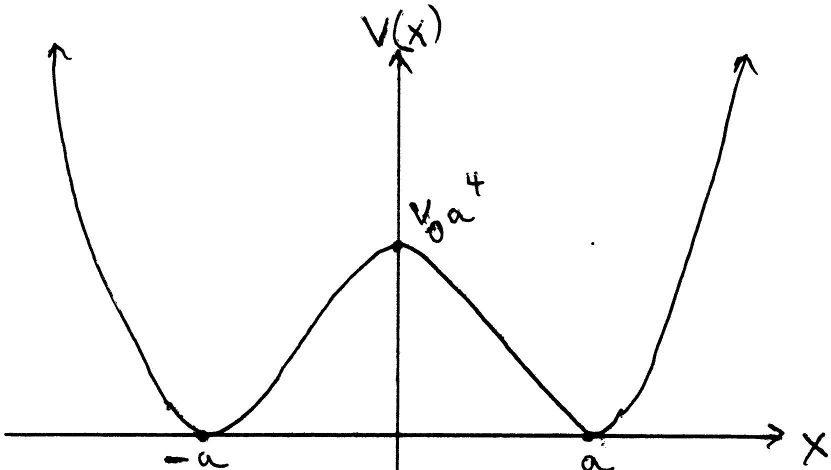


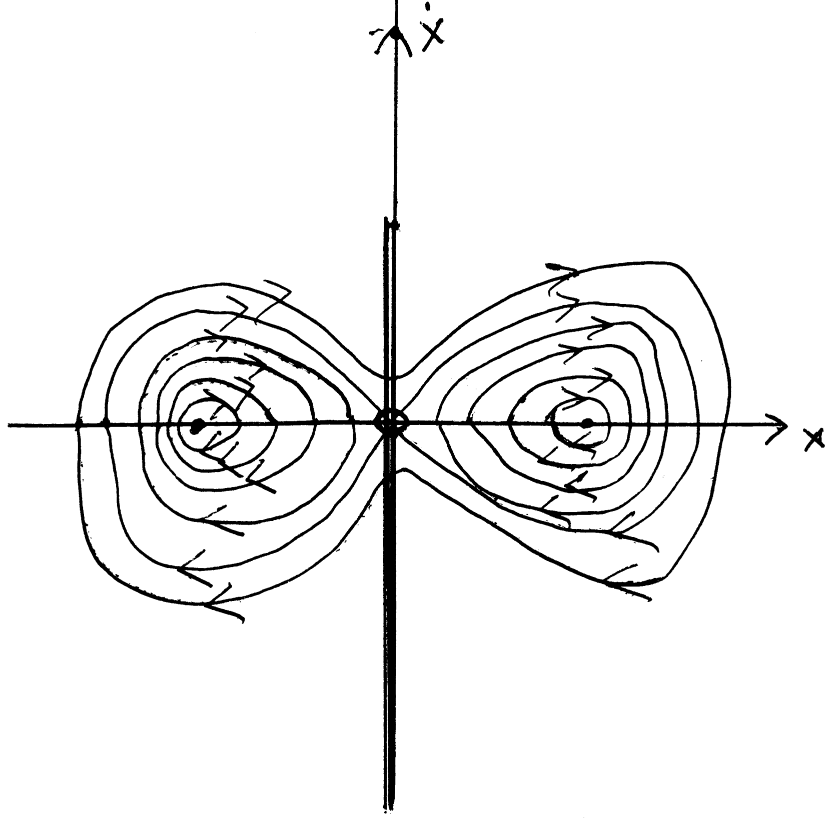
(b) ON THE ABOVE GRAPH, • ARE STABLE
⊙ ARE UNSTABLE

(c) SKETCH x(t) FOR SEVERAL INITIAL CONDITIONS



2 (a) $V(x) = V_0(a^2 - x^2)^2$

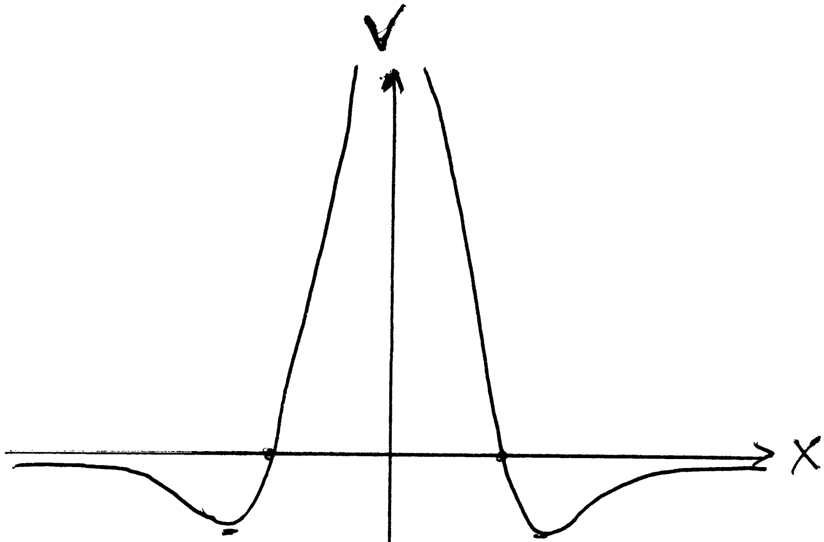


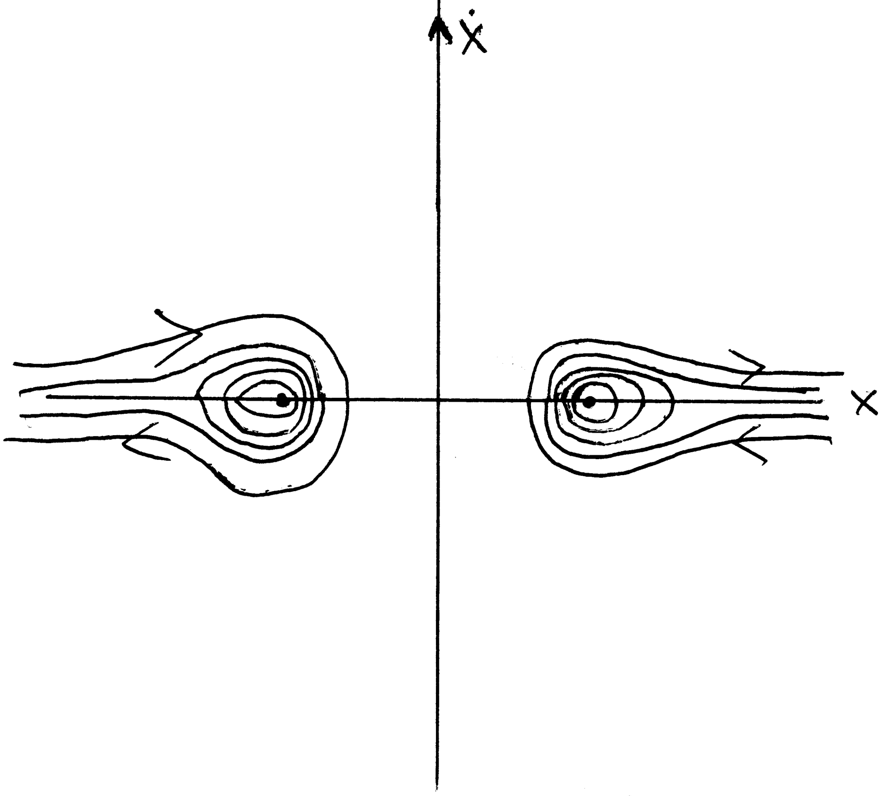


• STABLE

O UNSTABLE

2(b) $V(x) = V_0\left[\left(\frac{a}{x}\right)^4 - \left(\frac{a}{x}\right)^2\right]$





3. INTEGRATE $\dot{N} = rN\left(1 - \frac{N^2}{K^2}\right)$

$$\frac{dN}{dt} = rN\left(1 - \frac{N^2}{K^2}\right)$$

$$\frac{dN}{N\left(1 - \frac{N^2}{K^2}\right)} = r\,dt$$

SO, WE NEED TO SOLVE A PARTIAL FRACTION:

$$\frac{1}{N\left(1-\frac{N}{K}\right)\left(1+\frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}} + \frac{C}{1+\frac{N}{K}}$$

$$1 = A\left(1 - \frac{N^2}{K^2}\right) + B\left(N + \frac{N^2}{K}\right) + C\left(N - \frac{N^2}{K}\right)$$

N^0: $1 = A$

N^1: $0 = B + C$

N^2: $0 = -\frac{A}{K^2} + \frac{B}{K} - \frac{C}{K}$

$\therefore A = 1$

$B = \frac{1}{2K}$

$C = -\frac{1}{2K}$

SO, OUR DIF. EQ. BECOMES

$$\frac{dN}{N} + \frac{1}{2K}\frac{dN}{1-\frac{N}{K}} - \frac{1}{2K}\frac{dN}{1+\frac{N}{K}} = r\,dt$$

$$\ln N - \frac{1}{2}\ln\left(1 - \frac{N}{K}\right) - \frac{1}{2}\ln\left(1 + \frac{N}{K}\right) = rt + C$$

$$2 \ln N - \ln\left(1 - \frac{N^2}{K^2}\right) = 2rt + C$$

$$\frac{N^2}{1 - \frac{N^2}{k^2}} = Ce^{2rt}$$

$$\frac{1}{N^2} - \frac{1}{k^2} = Ce^{-2rt}$$

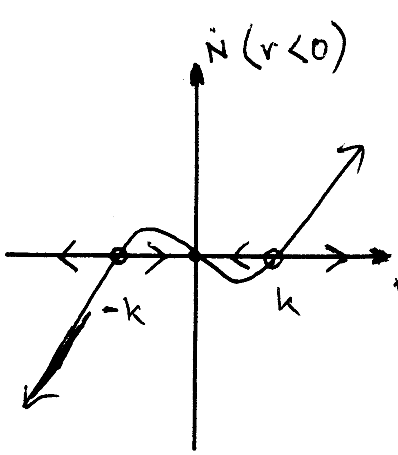
$$\frac{1}{N^2} = Ce^{-2rt} + \frac{1}{k^2}$$

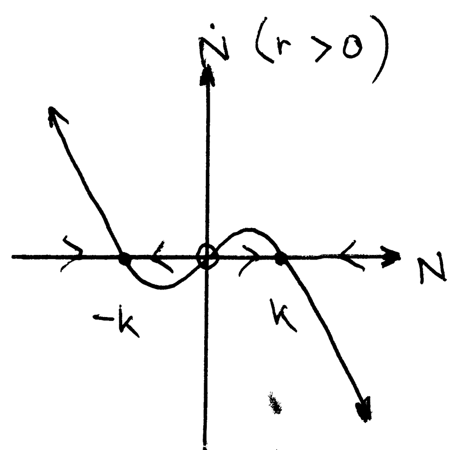
$$N = \left(Ce^{-2rt} + \frac{1}{k^2}\right)^{-1/2}$$

SO, SOLVING FOR $C(N_0)$ $\left(N_0 = N(t=0)\right)$

$$N = \left[\left(\frac{1}{N_0^2} - \frac{1}{k^2}\right)e^{-2rt} + \frac{1}{k^2}\right]^{-1/2}$$

PHASE SPACE





FOR $r<0$ AND $|N_0| > k$, FIND t_∞

$$N^{-2} = \left(\frac{1}{N_0^2} - \frac{1}{k^2}\right)e^{-2rt} + \frac{1}{k^2}$$

BUT $\lim_{N\to\infty} \frac{1}{N^2} = 0$

$$\therefore \quad 0 = \left(\frac{1}{N_0^2} - \frac{1}{k^2}\right)e^{-2rt_\infty} + \frac{1}{k^2}$$

$$-2rt_\infty = \ln \frac{-1}{\frac{k^2}{N_0^2} - 1}$$

$$-2rt_\infty = -\ln\left(\frac{-k^2}{N_0^2} + 1\right)$$

$$t_\infty = \frac{1}{2r} \ln\left(1 - \frac{k^2}{N_0^2}\right)$$

FOR $N_0^2 > k^2$, $\left(1 - \frac{k^2}{N_0^2}\right) \in [0, 1]$,

SO $\ln\left(1 - \frac{k^2}{N_0^2}\right)$ IS NEGATIVE, HENCE OUR NEED FOR $r<0$.

NOW, IF WE LOOK AT OUR SOLUTION FOR $N(t)$, WE CAN SEE THAT

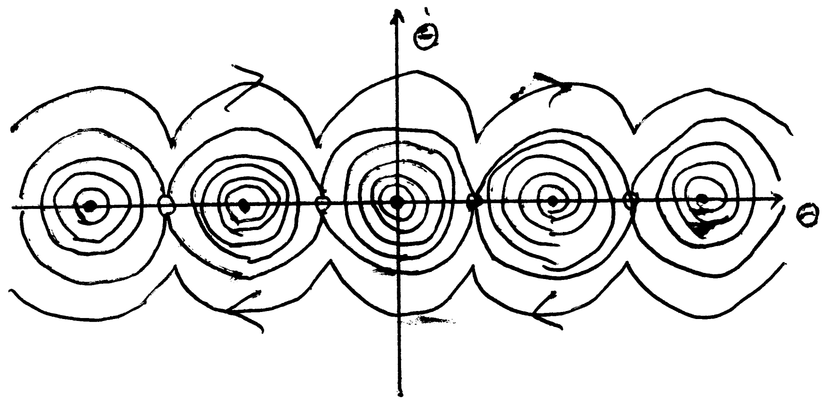
$$N^{-2} = x = \left(\frac{1}{N_0^2} - \frac{1}{k^2}\right)e^{-2rt} + \frac{1}{k^2}$$

IMPLIES THAT $N = \frac{1}{\sqrt{x}}$ WOULD MAKE A GOOD SUBSTITUTION IN OUR ORIGINAL DIFF. EQ.

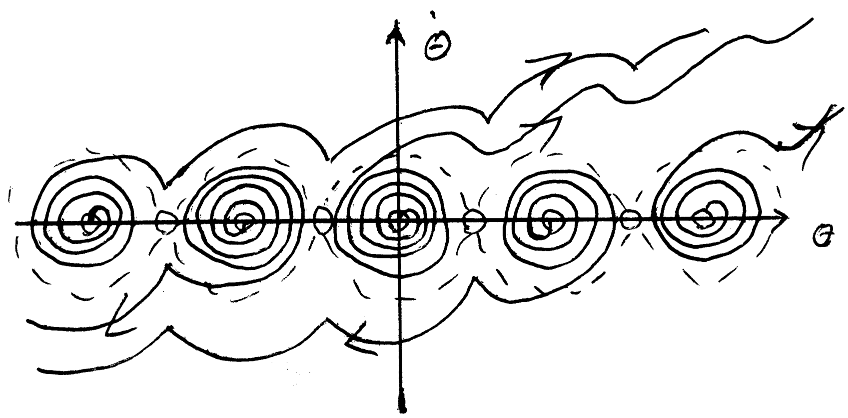
4) SKETCH THE PHASE CURVES FOR $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$

$$\vec{\varphi} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} \quad \dot{\vec{\varphi}} = \begin{pmatrix} \dot{\theta} \\ -b\dot{\theta} - \sin\theta \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -b\varphi_2 - \sin\varphi_1 \end{pmatrix}$$

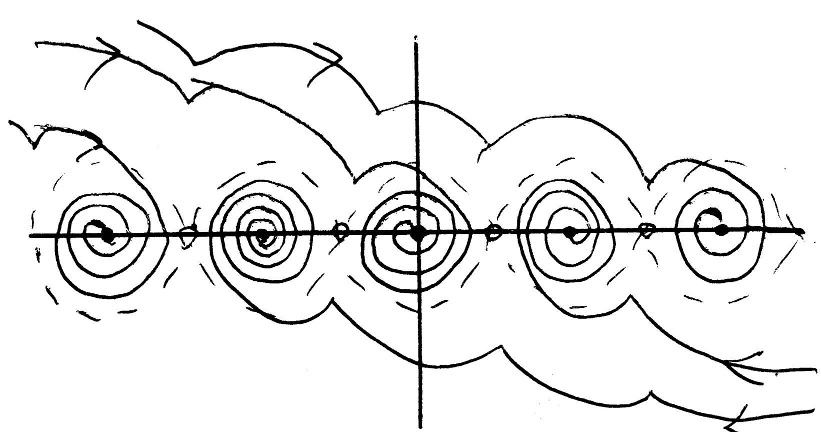
$b=0$



$b<0$



$b>0$



HERE, I HAVE CHOSEN TO DRAW THE ATTRACTORS/ REPULSORS IN ALTERNATING SIDES. BUT ALL OF THE ATTRACTORS

4 (CONT) ATTRACT EQUALLY FROM BOTH SIDES.

A GOOD PHYSICAL SYSTEM THAT OBEYS THESE EQUATIONS IS A DAMPED OR DRIVEN PENDULUM.

5) $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}$

WHERE $\vec{E} = E_y \hat{y} + E_z \hat{z}$

AND $\vec{B} = \vec{B}_z$

(a) FIND THE $\hat{z}$ MOTION. SOLVE FOR $z(t)$.

$$\begin{pmatrix} m\ddot{x} \\ m\ddot{y} \\ m\ddot{z} \end{pmatrix} = \begin{pmatrix} \frac{q}{c}B\dot{y} \\ qE_y - \frac{q}{c}B\dot{x} \\ qE_z \end{pmatrix}$$

SO, $\ddot{z} = \frac{q}{m}E_z$

OR, $z(t) = \frac{1}{2}\frac{q}{m}E_z t^2 + v_{0_z} t + z_0$

(b) $\ddot{x} = \frac{qB}{mc}\dot{y}$ (1)

$\ddot{y} = \frac{qE_y}{m} - \frac{qB}{mc}\dot{x}$ (2)

TAKE $\frac{d}{dt}$(2): $\dddot{y} = -\frac{qB}{mc}\ddot{x}$

SUBSTITUTE (1) INTO (2): $\dddot{y} = -\left(\frac{qB}{mc}\right)^2 \dot{y}$

SO, $\dddot{y} + \left(\frac{qB}{mc}\right)^2 \dot{y} = 0$

5 (CONT)

OR, $\ddot{v}_y + \omega_c v_y = 0$, WITH $\omega_c = \frac{qB}{mc}$ AND $\dot{y} = v_y$

THE EQUATION OF MOTION WE STARTED WITH IS THE GAUSSIAN VERSION. THEREFORE, WE WANT TO MEASURE B IN GAUSS, q IN esu, m IN g, AND c IN $cm \cdot s^{-1}$

$$\omega_c = \frac{qB}{mc}$$

$$\boxed{\omega_c = 1.05 \times 10^{12} \text{ rad/sec}}$$

$q = 4.8 \times 10^{-10}$ esu

$B = 6T = 6 \times 10^4$ GAUSS

$m_e = 9.11 \times 10^{-28}$ g

$c = 3.0 \times 10^{10}$ cm/sec

TO GET v_x, WE NOW TAKE A SIMILAR TRANSFORM:

$$\frac{d}{dt}(1): \quad \dddot{x} = \frac{qB}{mc}\ddot{y}$$

SUBSTITUTE (2) INTO (1): $\dddot{x} = \frac{q^2 B E_y}{m^2 c} - \frac{q^2 B^2}{m^2 c^2}\dot{x}$

OR $\ddot{v}_x + \omega_c^2 v_x = \omega_c \frac{qE_y}{m}$

WHICH IS A DISPLACED HARMONIC OSCILLATOR, ~~WHERE~~

~~$v_x = Ce^{i\omega_c t} + \frac{qE_y}{m\omega_c}$, WHERE C IS A CONSTANT OF INTEGRATION.~~

5 (CONT)

(c) $\dot{v}_y + \omega_c^2 v_y = 0$

WE KNOW THAT THE GENERAL SOLUTION TO THIS DIFF. EQ. IS $v_y = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$, FOR ARBITRARY C_1, C_2

$$\therefore \; v_y = C_1 e^{+i\omega_c t} + C_2 e^{-i\omega_c t} = \dot{y}$$

$$\dot{v}_x + \omega_c^2 v_x = \frac{\omega_c q E_y}{m}$$

THIS IS A DISPLACED HARMONIC OSCILLATOR. WE KNOW ITS SOLUTIONS WILL LOOK LIKE

$$v_x = C_3 e^{i\omega_c t} + C_4 e^{-i\omega_c t} + K,$$

WHERE K DETERMINES THE DISPLACEMENT

$$\therefore \; v_x = C_3 e^{i\omega_c t} + C_4 e^{-i\omega_c t} + \frac{qE_y}{m\omega_c} = \dot{x}$$

NOW, WE KNOW THAT $\dot{v}_x = \omega_c v_y$, AND $\dot{v}_y = \frac{qE_y}{m} - \omega_c v_x$

WHAT THIS TELLS ME IS THAT $\{C_1; C_2, C_3, C_4\}$ ARE NOT INDEPENDANT CONSTANTS.

$\dot{v}_x = \omega_c v_y$

$$i\omega_c C_3 e^{i\omega_c t} - i\omega_c C_4 e^{-i\omega_c t} = \omega_c C_1 e^{i\omega_c t} + \omega_c C_2 e^{-i\omega_c t}$$

AND $i\omega_c C_1 e^{i\omega_c t} - i\omega_c C_4 e^{-i\omega_c t} = \omega_c C_3 e^{i\omega_c t} + \omega_c C_4 e^{-i\omega_c t}$

OR, $iC_3 e^{i\omega_c t} - iC_4 e^{-i\omega_c t} = C_1 e^{i\omega_c t} + C_2 e^{-i\omega_c t}$

$iC_1 e^{i\omega_c t} - iC_2 e^{-i\omega_c t} = C_3 e^{i\omega_c t} + C_4 e^{-i\omega_c t}$

TAKE $t=0$: $C_1 + C_2 = iC_3 - iC_4$ OR, $C_1 + C_2 = i(C_3 - C_4)$

$iC_1 - iC_2 = C_3 + C_4$ —OR— $i(C_1 - C_2) = C_3 + C_4$

5(CONT)

$$iC_1 + iC_2 = C_4 - C_3$$

$$iC_1 - iC_2 = C_4 + C_3$$

$$2iC_1 = 2C_4$$

$$C_1 = -iC_4, \quad \text{AND} \quad C_2 = iC_3$$

$$\therefore v_x = -iC_2 e^{i\omega_c t} + iC_1 e^{-i\omega_c t} + \frac{qE_y}{m\omega_c}$$

$$v_y = C_1 e^{i\omega_c t} + C_2 e^{-i\omega_c t}$$

SO, TAKING $\dot{x} = v_x$ AND $\dot{y} = v_y$, WE INTEGRATE AND FIND

$$x(t) = -\frac{C_2}{\omega_c} e^{i\omega_c t} - \frac{C_1}{\omega_c} e^{-i\omega_c t} + \frac{qE_y}{m\omega_c} t + x_0$$

$$y(t) = \frac{C_1}{i\omega} e^{i\omega_c t} - \frac{C_2}{i\omega} e^{-i\omega_c t} + y_0$$

AND, FROM PART 1:

$$z(t) = \frac{1}{2}\frac{q}{m} E_z t^2 + v_{0_z} t + z_0$$

SO WE HAVE 6 CONSTANTS OF INTEGRATION.