Chapter 6

Lagrangian Mechanics

6.1 Generalized Coordinates

A set of *generalized coordinates* $q_1, \ldots, q_n$ completely describes the positions of all particles in a mechanical system. In a system with $d_{\rm f}$ degrees of freedom and k constraints, $n = d_{\rm f} - k$ independent generalized coordinates are needed to completely specify all the positions. A constraint is a relation among coordinates, such as $x^2 + y^2 + z^2 = a^2$ for a particle moving on a sphere of radius a. In this case, $d_{\rm f} = 3$ and $k = 1$. In this case, we could eliminate z in favor of x and y, *i.e.* by writing $z = \pm\sqrt{a^2 - x^2 - y^2}$, or we could choose as coordinates the polar and azimuthal angles θ and ϕ.

For the moment we will assume that $n = d_{\rm f} - k$, and that the generalized coordinates are independent, satisfying no additional constraints among them. Later on we will learn how to deal with any remaining constraints among the $\{q_1, \ldots, q_n\}$.

The generalized coordinates may have units of length, or angle, or perhaps something totally different. In the theory of small oscillations, the normal coordinates are conventionally chosen to have units of $(\text{mass})^{1/2} \times (\text{length})$. However, once a choice of generalized coordinate is made, with a concomitant set of units, the units of the conjugate momentum and force are determined:

$$\big[p_\sigma\big] = \frac{ML^2}{T} \cdot \frac{1}{\big[q_\sigma\big]} \qquad , \qquad \big[F_\sigma\big] = \frac{ML^2}{T^2} \cdot \frac{1}{\big[q_\sigma\big]} \ , \tag{6.1}$$

where $\big[A\big]$ means 'the units of A', and where M, L, and T stand for mass, length, and time, respectively. Thus, if q_σ has dimensions of length, then p_σ has dimensions of momentum and F_σ has dimensions of force. If q_σ is dimensionless, as is the case for an angle, p_σ has dimensions of angular momentum (ML^2/T) and F_σ has dimensions of torque (ML^2/T^2).

6.2 Hamilton's Principle

The equations of motion of classical mechanics are embodied in a variational principle, called *Hamilton's principle.* Hamilton's principle states that the motion of a system is such that the *action functional*

$$S\big[q(t)\big] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) \tag{6.2}$$

is an extremum, *i.e.* $\delta S = 0$. Here, $q = \{q_1, \ldots, q_n\}$ is a complete set of *generalized coordinates* for our mechanical system, and

$$L = T - U \tag{6.3}$$

is the *Lagrangian*, where T is the kinetic energy and U is the potential energy. Setting the first variation of the action to zero gives the Euler-Lagrange equations,

$$\frac{d}{dt} \overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}^{\text{momentum } p_\sigma} = \overbrace{\frac{\partial L}{\partial q_\sigma}}^{\text{force } F_\sigma} \quad . \tag{6.4}$$

Thus, we have the familiar $\dot{p}_\sigma = F_\sigma$, also known as Newton's second law. Note, however, that the $\{q_\sigma\}$ are *generalized coordinates*, so p_σ may not have dimensions of momentum, nor F_σ of force. For example, if the generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

6.2.1 Invariance of the equations of motion

Suppose

$$\tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}\, G(q, t) \ . \tag{6.5}$$

Then

$$\tilde{S}[q(t)] = S[q(t)] + G(q_b, t_b) - G(q_a, t_a) \ . \tag{6.6}$$

Since the difference $\tilde{S} - S$ is a function only of the endpoint values $\{q_a, q_b\}$, their variations are identical: $\delta\tilde{S} = \delta S$. This means that L and $\tilde{L}$ result in the same equations of motion. Thus, the equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time.

6.2.2 Remarks on the order of the equations of motion

The equations of motion are second order in time. This follows from the fact that $L = L(q, \dot{q}, t)$. Using the chain rule,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial \dot{q}_{\sigma'}}\, \ddot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial q_{\sigma'}}\, \dot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial t} \ . \tag{6.7}$$

That the equations are second order in time can be regarded as an empirical fact. It follows, as we have just seen, from the fact that L depends on q and on $\dot{q}$, but on no higher time derivative terms. Suppose the Lagrangian did depend on the generalized accelerations $\ddot{q}$ as well. What would the equations of motion look like?

Taking the variation of S,

$$\delta\int_{t_a}^{t_b} dt\, L(q,\dot{q},\ddot{q},t) = \left[\frac{\partial L}{\partial \dot{q}_\sigma}\,\delta q_\sigma + \frac{\partial L}{\partial \ddot{q}_\sigma}\,\delta\dot{q}_\sigma - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\delta q_\sigma\right]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\right\}\delta q_\sigma \ . \tag{6.8}$$

The boundary term vanishes if we require $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = \delta\dot{q}_\sigma(t_a) = \delta\dot{q}_\sigma(t_b) = 0 \ \forall\ \sigma$. The equations of motion would then be *fourth order* in time.

6.2.3 Lagrangian for a free particle

For a free particle, we can use Cartesian coordinates for each particle as our system of generalized coordinates. For a single particle, the Lagrangian $L(\boldsymbol{x},\boldsymbol{v},t)$ must be a function solely of $\boldsymbol{v}^2$. This is because homogeneity with respect to space and time preclude any dependence of L on $\boldsymbol{x}$ or on t, and isotropy of space means L must depend on $\boldsymbol{v}^2$. We next invoke Galilean relativity, which says that the equations of motion are invariant under transformation to a reference frame moving with constant velocity. Let $\boldsymbol{V}$ be the velocity of the new reference frame $\mathcal{K}'$ relative to our initial reference frame $\mathcal{K}$. Then $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V}t$, and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}$. In order that the equations of motion be invariant under the change in reference frame, we demand

$$L'(\boldsymbol{v}') = L(\boldsymbol{v}) + \frac{d}{dt}\,G(\boldsymbol{x},t) \ . \tag{6.9}$$

The only possibility is $L = \frac{1}{2}m\boldsymbol{v}^2$, where the constant m is the mass of the particle. Note:

$$L' = \tfrac{1}{2}m(\boldsymbol{v}-\boldsymbol{V})^2 = \tfrac{1}{2}m\boldsymbol{v}^2 + \frac{d}{dt}\Big(\tfrac{1}{2}m\boldsymbol{V}^2\,t - m\boldsymbol{V}\cdot\boldsymbol{x}\Big) = L + \frac{dG}{dt} \ . \tag{6.10}$$

For N interacting particles,

$$L = \tfrac{1}{2}\sum_{a=1}^{N} m_a \left(\frac{d\boldsymbol{x}_a}{dt}\right)^2 - U\big(\{\boldsymbol{x}_a\},\{\dot{\boldsymbol{x}}_a\}\big) \ . \tag{6.11}$$

Here, U is the *potential energy*. Generally, U is of the form

$$U = \sum_a U_1(\boldsymbol{x}_a) + \sum_{a<a'} v(\boldsymbol{x}_a - \boldsymbol{x}_{a'}) \ , \tag{6.12}$$

however, as we shall see, velocity-dependent potentials appear in the case of charged particles interacting with electromagnetic fields. In general, though,

$$L = T - U \ , \tag{6.13}$$

where T is the kinetic energy, and U is the potential energy.

6.3 Conserved Quantities

A conserved quantity $\Lambda(q, \dot{q}, t)$ is one which does not vary throughout the motion of the system. This means

$$\frac{d\Lambda}{dt}\bigg|_{q=q(t)} = 0 \ . \tag{6.14}$$

We shall discuss conserved quantities in detail in the chapter on Noether's Theorem, which follows.

6.3.1 Momentum conservation

The simplest case of a conserved quantity occurs when the Lagrangian does not explicitly depend on one or more of the generalized coordinates, *i.e.* when

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = 0 \ . \tag{6.15}$$

We then say that L is *cyclic* in the coordinate q_σ. In this case, the Euler-Lagrange equations $\dot{p}_\sigma = F_\sigma$ say that the conjugate momentum p_σ is conserved. Consider, for example, the motion of a particle of mass m near the surface of the earth. Let (x, y) be coordinates parallel to the surface and z the height. We then have

$$T = \tfrac{1}{2} m \big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) \tag{6.16}$$

$$U = mgz \tag{6.17}$$

$$L = T - U = \tfrac{1}{2} m \big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) - mgz \ . \tag{6.18}$$

Since

$$F_x = \frac{\partial L}{\partial x} = 0 \quad \text{and} \quad F_y = \frac{\partial L}{\partial y} = 0 \ , \tag{6.19}$$

we have that p_x and p_y are conserved, with

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \ . \tag{6.20}$$

These first order equations can be integrated to yield

$$x(t) = x(0) + \frac{p_x}{m}\, t \quad , \quad y(t) = y(0) + \frac{p_y}{m}\, t \ . \tag{6.21}$$

The z equation is of course

$$\dot{p}_z = m\ddot{z} = -mg = F_z \ , \tag{6.22}$$

with solution

$$z(t) = z(0) + \dot{z}(0)\, t - \tfrac{1}{2} g t^2 \ . \tag{6.23}$$

As another example, consider a particle moving in the (x, y) plane under the influence of a potential $U(x, y) = U\big(\sqrt{x^2 + y^2}\big)$ which depends only on the particle's distance from the origin $\rho = \sqrt{x^2 + y^2}$. The Lagrangian, expressed in two-dimensional polar coordinates (ρ, ϕ), is

$$L = \tfrac{1}{2} m \big(\dot{\rho}^2 + \rho^2 \dot{\phi}^2\big) - U(\rho) \ . \tag{6.24}$$

We see that L is cyclic in the angle ϕ, hence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} \tag{6.25}$$

is conserved. p_ϕ is the angular momentum of the particle about the $\hat{\boldsymbol{z}}$ axis. In the language of the calculus of variations, momentum conservation is what follows when the integrand of a functional is independent of the *independent variable*.

6.3.2 Energy conservation

When the integrand of a functional is independent of the *dependent* variable, another conservation law follows. For Lagrangian mechanics, consider the expression

$$H(q, \dot{q}, t) = \sum_{\sigma=1}^{n} p_\sigma \, \dot{q}_\sigma - L \ . \tag{6.26}$$

Now we take the total time derivative of H:

$$\frac{dH}{dt} = \sum_{\sigma=1}^{n} \left\{ p_\sigma \, \ddot{q}_\sigma + \dot{p}_\sigma \, \dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} \, \dot{q}_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} \, \ddot{q}_\sigma \right\} - \frac{\partial L}{\partial t} \ . \tag{6.27}$$

We evaluate $\dot{H}$ along the motion of the system, which entails that the terms in the curly brackets above cancel for each σ:

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad \dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma} \ . \tag{6.28}$$

Thus, we find

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{6.29}$$

which means that H is conserved *whenever the Lagrangian contains no explicit time dependence*. For a Lagrangian of the form

$$L = \sum_a \tfrac{1}{2} m_a \dot{\boldsymbol{r}}_a^2 - U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \ , \tag{6.30}$$

we have that $\boldsymbol{p}_a = m_a\,\dot{\boldsymbol{r}}_a$, and

$$H = T + U = \sum_a \tfrac{1}{2}\,m_a \dot{\boldsymbol{r}}_a^2 + U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)\ . \tag{6.31}$$

However, it is not always the case that $H = T + U$ is the total energy, as we shall see in the next chapter.

6.4 Choosing Generalized Coordinates

Any choice of generalized coordinates will yield an equivalent set of equations of motion. However, some choices result in an apparently simpler set than others. This is often true with respect to the form of the potential energy. Additionally, certain constraints that may be present are more amenable to treatment using a particular set of generalized coordinates.

The kinetic energy T is always simple to write in Cartesian coordinates, and it is good practice, at least when one is first learning the method, to write T in Cartesian coordinates and then convert to generalized coordinates. In Cartesian coordinates, the kinetic energy of a single particle of mass m is

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2\right)\ . \tag{6.32}$$

If the motion is two-dimensional, and confined to the plane $z =$ const., one of course has $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$.

Two other commonly used coordinate systems are the cylindrical and spherical systems. In cylindrical coordinates (ρ, ϕ, z), ρ is the radial coordinate in the (x, y) plane and ϕ is the azimuthal angle:

$$x = \rho\,\cos\phi \qquad\qquad \dot{x} = \cos\phi\,\dot{\rho} - \rho\sin\phi\,\dot{\phi} \tag{6.33}$$

$$y = \rho\,\sin\phi \qquad\qquad \dot{y} = \sin\phi\,\dot{\rho} + \rho\cos\phi\,\dot{\phi}\ , \tag{6.34}$$

and the third, orthogonal coordinate is of course z. The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2\right) \\ &= \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2 + \dot{z}^2\right)\ . \end{aligned} \tag{6.35}$$

When the motion is confined to a plane with $z =$ const., this coordinate system is often referred to as 'two-dimensional polar' coordinates.

In spherical coordinates (r, θ, ϕ), r is the radius, θ is the polar angle, and ϕ is the azimuthal angle. On the globe, θ would be the 'colatitude', which is $\theta = \frac{\pi}{2} - \lambda$, where λ is the latitude. *I.e.* $\theta = 0$ at the north pole. In spherical polar coordinates,

$$x = r\,\sin\theta\,\cos\phi \qquad \dot{x} = \sin\theta\,\cos\phi\,\dot{r} + r\,\cos\theta\,\cos\phi\,\dot{\theta} - r\,\sin\theta\sin\phi\,\dot{\phi} \tag{6.36}$$

$$y = r\,\sin\theta\,\sin\phi \qquad \dot{y} = \sin\theta\,\sin\phi\,\dot{r} + r\,\cos\theta\,\sin\phi\,\dot{\theta} + r\,\sin\theta\cos\phi\,\dot{\phi} \tag{6.37}$$

$$z = r\,\cos\theta \qquad \dot{z} = \cos\theta\,\dot{r} - r\,\sin\theta\,\dot{\theta}\ . \tag{6.38}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \\ &= \tfrac{1}{2}m\left(\dot{r}^2+r^2\,\dot{\theta}^2+r^2\,\sin^2\!\theta\,\dot{\phi}^2\right)\ . \end{aligned} \tag{6.39}$$

6.5 How to Solve Mechanics Problems

Here are some simple steps you can follow toward obtaining the equations of motion:

1. Choose a set of generalized coordinates $\{q_1,\ldots,q_n\}$.
2. Find the kinetic energy $T(q,\dot{q},t)$, the potential energy $U(q,t)$, and the Lagrangian $L(q,\dot{q},t)=T-U$. It is often helpful to first write the kinetic energy in Cartesian coordinates for each particle before converting to generalized coordinates.
3. Find the canonical momenta $p_\sigma=\frac{\partial L}{\partial\dot{q}_\sigma}$ and the generalized forces $F_\sigma=\frac{\partial L}{\partial q_\sigma}$.
4. Evaluate the time derivatives $\dot{p}_\sigma$ and write the equations of motion $\dot{p}_\sigma=F_\sigma$. Be careful to differentiate properly, using the chain rule and the Leibniz rule where appropriate.
5. Identify any conserved quantities (more about this later).

6.6 Examples

6.6.1 One-dimensional motion

For a one-dimensional mechanical system with potential energy $U(x)$,

$$L=T-U=\tfrac{1}{2}m\dot{x}^2-U(x)\ . \tag{6.40}$$

The canonical momentum is

$$p=\frac{\partial L}{\partial\dot{x}}=m\dot{x} \tag{6.41}$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)=\frac{\partial L}{\partial x}\qquad\Rightarrow\qquad m\ddot{x}=-U'(x)\ , \tag{6.42}$$

which is of course $F=ma$.

Note that we can multiply the equation of motion by $\dot{x}$ to get

$$0=\dot{x}\,\Big\{m\ddot{x}+U'(x)\Big\}=\frac{d}{dt}\Big\{\tfrac{1}{2}m\dot{x}^2+U(x)\Big\}=\frac{dE}{dt}\ , \tag{6.43}$$

where $E=T+U$.

6.6.2 Central force in two dimensions

Consider next a particle of mass m moving in two dimensions under the influence of a potential $U(\rho)$ which is a function of the distance from the origin $\rho = \sqrt{x^2 + y^2}$. Clearly cylindrical ($2d$ polar) coordinates are called for:

$$L = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2\right) - U(\rho)\ . \tag{6.44}$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = \frac{\partial L}{\partial \rho} \quad \Rightarrow \quad m\ddot{\rho} = m\rho\,\dot{\phi}^2 - U'(\rho) \tag{6.45}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad \Rightarrow \quad \frac{d}{dt}\left(m\rho^2\,\dot{\phi}\right) = 0\ . \tag{6.46}$$

Note that the canonical momentum conjugate to ϕ, which is to say the angular momentum, is conserved:

$$p_\phi = m\rho^2\,\dot{\phi} = \text{const.} \tag{6.47}$$

We can use this to eliminate $\dot{\phi}$ from the first Euler-Lagrange equation, obtaining

$$m\ddot{\rho} = \frac{p_\phi^2}{m\rho^3} - U'(\rho)\ . \tag{6.48}$$

We can also write the total energy as

$$\begin{aligned} E &= \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2\right) + U(\rho) \\ &= \tfrac{1}{2}m\,\dot{\rho}^2 + \frac{p_\phi^2}{2m\rho^2} + U(\rho)\ , \end{aligned} \tag{6.49}$$

from which it may be shown that E is also a constant:

$$\frac{dE}{dt} = \left(m\,\ddot{\rho} - \frac{p_\phi^2}{m\rho^3} + U'(\rho)\right)\dot{\rho} = 0\ . \tag{6.50}$$

We shall discuss this case in much greater detail in the coming weeks.

6.6.3 A sliding point mass on a sliding wedge

Consider the situation depicted in Fig. 6.1, in which a point object of mass m slides frictionlessly along a wedge of opening angle α. The wedge itself slides frictionlessly along a horizontal surface, and its mass is M. We choose as generalized coordinates the horizontal position X of the left corner of the wedge, and the horizontal distance x from the left corner to the sliding point mass. The vertical coordinate of the sliding mass is then $y = x\,\tan\alpha$,

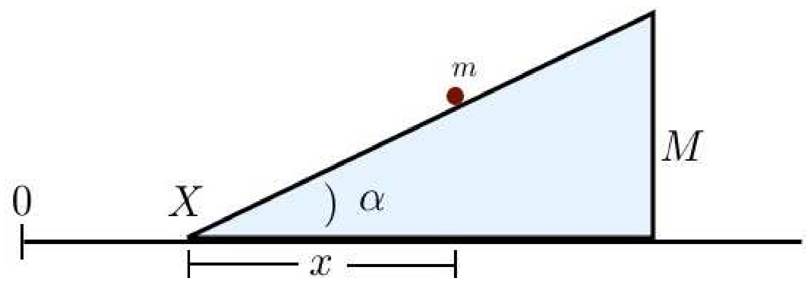

Figure 6.1: A wedge of mass M and opening angle α slides frictionlessly along a horizontal surface, while a small object of mass m slides frictionlessly along the wedge.

where the horizontal surface lies at $y = 0$. With these generalized coordinates, the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\,(\dot{X}+\dot{x})^2 + \tfrac{1}{2}m\dot{y}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot{x}^2\ . \end{aligned} \tag{6.51}$$

The potential energy is simply

$$U = mgy = mg\,x\,\tan\alpha\ . \tag{6.52}$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot{x}^2 - mg\,x\,\tan\alpha\ , \tag{6.53}$$

and the equations of motion are

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{X}}\bigg) &= \frac{\partial L}{\partial X} \qquad \Rightarrow \qquad (M+m)\ddot{X} + m\,\ddot{x} = 0 \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{x}}\bigg) &= \frac{\partial L}{\partial x} \qquad \Rightarrow \qquad m\ddot{X} + m\,(1+\tan^2\!\alpha)\,\ddot{x} = -mg\tan\alpha\ . \end{aligned} \tag{6.54}$$

At this point we can use the first of these equations to write

$$\ddot{X} = -\frac{m}{M+m}\,\ddot{x}\ . \tag{6.55}$$

Substituting this into the second equation, we obtain the constant accelerations

$$\ddot{x} = -\frac{(M+m)g\,\sin\alpha\,\cos\alpha}{M+m\sin^2\!\alpha} \quad , \quad \ddot{X} = \frac{mg\,\sin\alpha\,\cos\alpha}{M+m\sin^2\!\alpha}\ . \tag{6.56}$$

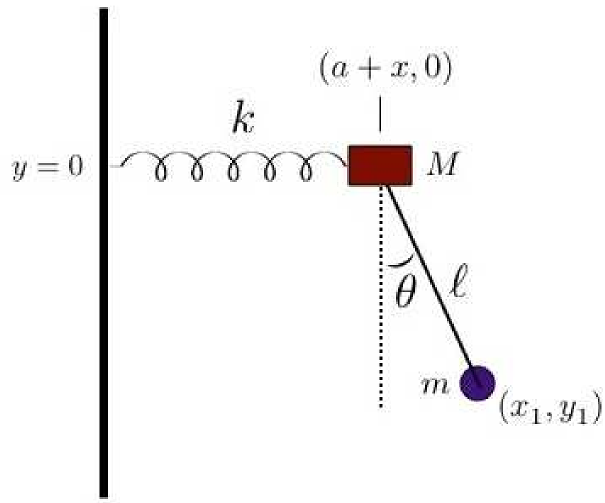

Figure 6.2: The spring–pendulum system.

6.6.4 A pendulum attached to a mass on a spring

Consider next the system depicted in Fig. 6.2 in which a mass M moves horizontally while attached to a spring of spring constant k. Hanging from this mass is a pendulum of arm length ℓ and bob mass m.

A convenient set of generalized coordinates is (x, θ), where x is the displacement of the mass M relative to the equilibrium extension a of the spring, and θ is the angle the pendulum arm makes with respect to the vertical. Let the Cartesian coordinates of the pendulum bob be (x_1, y_1). Then

$$x_1 = a + x + \ell \sin\theta \quad , \quad y_1 = -l \cos\theta \ . \tag{6.57}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2) \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\Big[(\dot{x} + \ell\cos\theta\,\dot{\theta})^2 + (\ell\sin\theta\,\dot{\theta})^2\Big] \\ &= \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta} \ , \end{aligned} \tag{6.58}$$

and the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kx^2 + mgy_1 \\ &= \tfrac{1}{2}kx^2 - mg\ell\cos\theta \ . \end{aligned} \tag{6.59}$$

Thus,

$$L = \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta} - \tfrac{1}{2}kx^2 + mg\ell\cos\theta \ . \tag{6.60}$$

The canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\,\dot{x} + m\ell\cos\theta\,\dot{\theta}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell\cos\theta\,\dot{x} + m\ell^2\,\dot{\theta}\ , \tag{6.61}$$

and the canonical forces are

$$F_x = \frac{\partial L}{\partial x} = -kx$$

$$F_\theta = \frac{\partial L}{\partial \theta} = -m\ell\sin\theta\,\dot{x}\,\dot{\theta} - mg\ell\,\sin\theta\ . \tag{6.62}$$

The equations of motion then yield

$$(M+m)\,\ddot{x} + m\ell\,\cos\theta\,\ddot{\theta} - m\ell\,\sin\theta\,\dot{\theta}^2 = -kx \tag{6.63}$$

$$m\ell\,\cos\theta\,\ddot{x} + m\ell^2\,\ddot{\theta} = -mg\ell\,\sin\theta\ . \tag{6.64}$$

Small Oscillations : If we assume both x and θ are small, we may write $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, in which case the equations of motion may be linearized to

$$(M+m)\,\ddot{x} + m\ell\,\ddot{\theta} + kx = 0 \tag{6.65}$$

$$m\ell\,\ddot{x} + m\ell^2\,\ddot{\theta} + mg\ell\,\theta = 0\ . \tag{6.66}$$

If we define

$$u \equiv \frac{x}{\ell} \quad , \quad \alpha \equiv \frac{m}{M} \quad , \quad \omega_0^2 \equiv \frac{k}{M} \quad , \quad \omega_1^2 \equiv \frac{g}{\ell}\ , \tag{6.67}$$

then

$$(1+\alpha)\,\ddot{u} + \alpha\,\ddot{\theta} + \omega_0^2\,u = 0 \tag{6.68}$$

$$\ddot{u} + \ddot{\theta} + \omega_1^2\,\theta = 0\ . \tag{6.69}$$

We can solve by writing

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-i\omega t}\ , \tag{6.70}$$

in which case

$$\begin{pmatrix} \omega_0^2 - (1+\alpha)\,\omega^2 & -\alpha\,\omega^2 \\ -\omega^2 & \omega_1^2 - \omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ . \tag{6.71}$$

In order to have a nontrivial solution (*i.e.* without $a=b=0$), the determinant of the above 2×2 matrix must vanish. This gives a condition on ω^2, with solutions

$$\omega_\pm^2 = \tfrac{1}{2}\big(\omega_0^2 + (1+\alpha)\,\omega_1^2\big) \pm \tfrac{1}{2}\sqrt{\big(\omega_0^2-\omega_1^2\big)^2 + 2\alpha\,(\omega_0^2+\omega_1^2)\,\omega_1^2}\ . \tag{6.72}$$

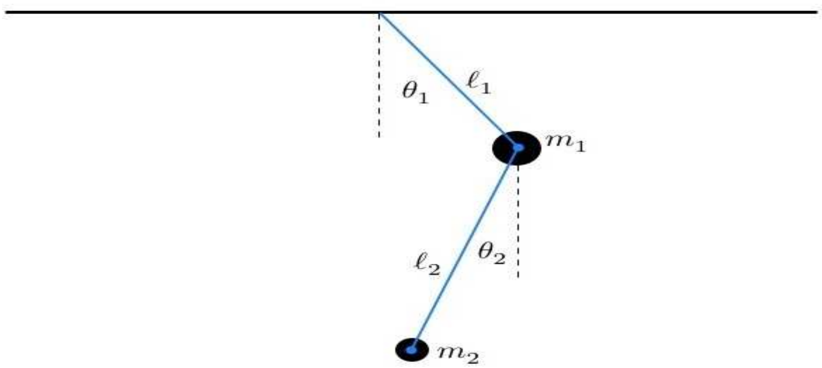

Figure 6.3: The double pendulum, with generalized coordinates θ_1 and θ_2. All motion is confined to a single plane.

6.6.5 The double pendulum

As yet another example of the generalized coordinate approach to Lagrangian dynamics, consider the double pendulum system, sketched in Fig. 6.3. We choose as generalized coordinates the two angles θ_1 and θ_2. In order to evaluate the Lagrangian, we must obtain the kinetic and potential energies in terms of the generalized coordinates $\{\theta_1, \theta_2\}$ and their corresponding velocities $\{\dot{\theta}_1, \dot{\theta}_2\}$.

In Cartesian coordinates,

$$T = \tfrac{1}{2}m_1\,(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2\,(\dot{x}_2^2 + \dot{y}_2^2) \tag{6.73}$$

$$U = m_1\,g\,y_1 + m_2\,g\,y_2\ . \tag{6.74}$$

We therefore express the Cartesian coordinates $\{x_1, y_1, x_2, y_2\}$ in terms of the generalized coordinates $\{\theta_1, \theta_2\}$:

$$x_1 = \ell_1\,\sin\theta_1 \qquad\qquad x_2 = \ell_1\,\sin\theta_1 + \ell_2\,\sin\theta_2 \tag{6.75}$$

$$y_1 = -\ell_1\,\cos\theta_1 \qquad\qquad y_2 = -\ell_1\,\cos\theta_1 - \ell_2\,\cos\theta_2\ . \tag{6.76}$$

Thus, the velocities are

$$\dot{x}_1 = \ell_1\,\dot{\theta}_1\,\cos\theta_1 \qquad\qquad \dot{x}_2 = \ell_1\,\dot{\theta}_1\,\cos\theta_1 + \ell_2\,\dot{\theta}_2\,\cos\theta_2 \tag{6.77}$$

$$\dot{y}_1 = \ell_1\,\dot{\theta}_1\,\sin\theta_1 \qquad\qquad \dot{y}_2 = \ell_1\,\dot{\theta}_1\,\sin\theta_1 + \ell_2\,\dot{\theta}_2\,\sin\theta_2\ . \tag{6.78}$$

Thus,

$$T = \tfrac{1}{2}m_1\,\ell_1^2\,\dot{\theta}_1^2 + \tfrac{1}{2}m_2\Big\{\ell_1^2\,\dot{\theta}_1^2 + 2\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 + \ell_2^2\,\dot{\theta}_2^2\Big\} \tag{6.79}$$

$$U = -m_1\,g\,\ell_1\,\cos\theta_1 - m_2\,g\,\ell_1\,\cos\theta_1 - m_2\,g\,\ell_2\,\cos\theta_2\ , \tag{6.80}$$

and

$$
\begin{aligned}
L = T - U = \tfrac{1}{2}(m_1 + m_2)\,\ell_1^2\,\dot\theta_1^2 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 + \tfrac{1}{2} m_2\,\ell_2^2\,\dot\theta_2^2 \\
+ (m_1 + m_2)\,g\,\ell_1\,\cos\theta_1 + m_2\,g\,\ell_2\,\cos\theta_2\ . \qquad (6.81)
\end{aligned}
$$

The generalized (canonical) momenta are

$$p_1 = \frac{\partial L}{\partial\dot\theta_1} = (m_1 + m_2)\,\ell_1^2\,\dot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_2 \qquad (6.82)$$

$$p_2 = \frac{\partial L}{\partial\dot\theta_2} = m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\dot\theta_2\ , \qquad (6.83)$$

and the equations of motion are

$$
\begin{aligned}
\dot p_1 &= (m_1 + m_2)\,\ell_1^2\,\ddot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\ddot\theta_2 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,(\dot\theta_1 - \dot\theta_2)\,\dot\theta_2 \\
&= -(m_1 + m_2)\,g\,\ell_1\,\sin\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_1} \qquad (6.84)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot p_2 &= m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\ddot\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,(\dot\theta_1 - \dot\theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\ddot\theta_2 \\
&= -m_2\,g\,\ell_2\,\sin\theta_2 + m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_2}\ . \qquad (6.85)
\end{aligned}
$$

We therefore find

$$\ell_1\,\ddot\theta_1 + \frac{m_2\,\ell_2}{m_1 + m_2}\,\cos(\theta_1 - \theta_2)\,\ddot\theta_2 + \frac{m_2\,\ell_2}{m_1 + m_2}\,\sin(\theta_1 - \theta_2)\,\dot\theta_2^2 + g\,\sin\theta_1 = 0 \qquad (6.86)$$

$$\ell_1\,\cos(\theta_1 - \theta_2)\,\ddot\theta_1 + \ell_2\,\ddot\theta_2 - \ell_1\,\sin(\theta_1 - \theta_2)\,\dot\theta_1^2 + g\,\sin\theta_2 = 0\ . \qquad (6.87)$$

Small Oscillations : The equations of motion are coupled, nonlinear second order ODEs. When the system is close to equilibrium, the amplitudes of the motion are small, and we may expand in powers of the θ_1 and θ_2. The linearized equations of motion are then

$$\ddot\theta_1 + \alpha\beta\,\ddot\theta_2 + \omega_0^2\,\theta_1 = 0 \qquad (6.88)$$

$$\ddot\theta_1 + \beta\,\ddot\theta_2 + \omega_0^2\,\theta_2 = 0\ , \qquad (6.89)$$

where we have defined

$$\alpha \equiv \frac{m_2}{m_1 + m_2} \quad , \quad \beta \equiv \frac{\ell_2}{\ell_1} \quad , \quad \omega_0^2 \equiv \frac{g}{\ell_1}\ . \qquad (6.90)$$

We can solve this coupled set of equations by a nifty trick. Let's take a linear combination of the first equation plus an undetermined coefficient, r, times the second:

$$(1+r)\,\ddot{\theta}_1 + (\alpha+r)\beta\,\ddot{\theta}_2 + \omega_0^2\,(\theta_1 + r\,\theta_2) = 0\ . \tag{6.91}$$

We now demand that the ratio of the coefficients of θ_2 and θ_1 is the same as the ratio of the coefficients of $\ddot{\theta}_2$ and $\ddot{\theta}_1$:

$$\frac{(\alpha+r)\,\beta}{1+r} = r \qquad \Rightarrow \qquad r_\pm = \tfrac{1}{2}(\beta-1) \pm \tfrac{1}{2}\sqrt{(1-\beta)^2 + 4\alpha\beta} \tag{6.92}$$

When $r = r_\pm$, the equation of motion may be written

$$\frac{d^2}{dt^2}\big(\theta_1 + r_\pm\,\theta_2\big) = -\frac{\omega_0^2}{1+r_\pm}\big(\theta_1 + r_\pm\,\theta_2\big) \tag{6.93}$$

and defining the (unnormalized) *normal modes*

$$\xi_\pm \equiv \big(\theta_1 + r_\pm\,\theta_2\big)\ , \tag{6.94}$$

we find

$$\ddot{\xi}_\pm + \omega_\pm^2\,\xi_\pm = 0\ , \tag{6.95}$$

with

$$\omega_\pm = \frac{\omega_0}{\sqrt{1+r_\pm}}\ . \tag{6.96}$$

Thus, by switching to the normal coordinates, we decoupled the equations of motion, and identified the two *normal frequencies of oscillation*. We shall have much more to say about small oscillations further below.

For example, with $\ell_1 = \ell_2 = \ell$ and $m_1 = m_2 = m$, we have $\alpha = \frac{1}{2}$, and $\beta = 1$, in which case

$$r_\pm = \pm\tfrac{1}{\sqrt{2}} \quad , \quad \xi_\pm = \theta_1 \pm \tfrac{1}{\sqrt{2}}\,\theta_2 \quad , \quad \omega_\pm = \sqrt{2 \mp \sqrt{2}}\,\sqrt{\frac{g}{\ell}}\ . \tag{6.97}$$

Note that the oscillation frequency for the 'in-phase' mode ξ_+ is low, and that for the 'out of phase' mode ξ_- is high.

6.6.6 The thingy

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached to each vertex. The opposite corners are joined by springs of spring constant k. In the square configuration, the strings are unstretched. The motion is confined to a plane, and the particles move only along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about equilibrium.

Solution

The rhombus is depicted in figure 6.4. Let a be the equilibrium length of the springs; clearly $L = \frac{a}{\sqrt{2}}$. Let ϕ be half of one of the opening angles, as shown. Then the masses are located at $(\pm X, 0)$ and $(0, \pm Y)$, with $X = \frac{a}{\sqrt{2}} \cos\phi$ and $Y = \frac{a}{\sqrt{2}} \sin\phi$. The spring extensions are $\delta X = 2X - a$ and $\delta Y = 2Y - a$. The kinetic and potential energies are therefore

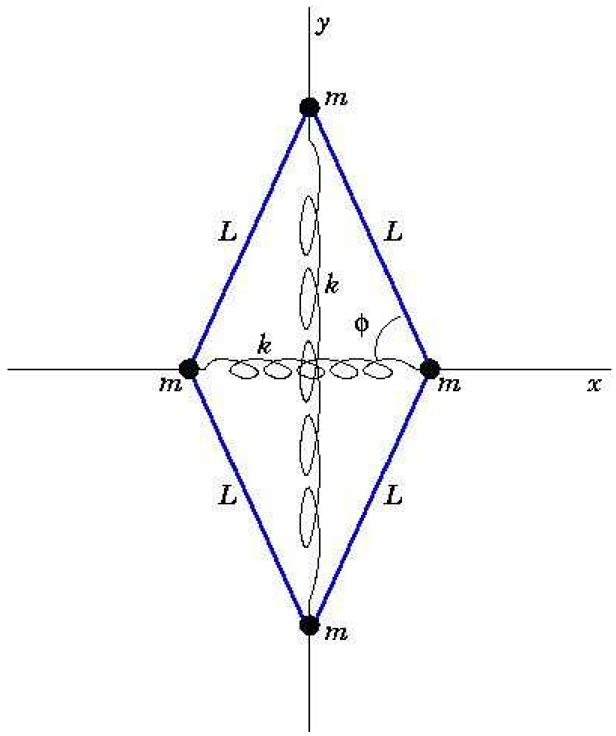

Figure 6.4: The thingy: a rhombus with opening angles 2ϕ and $\pi - 2\phi$.

$$\begin{aligned} T &= M\big(\dot{X}^2 + \dot{Y}^2\big) \\ &= \tfrac{1}{2} M a^2 \, \dot{\phi}^2 \end{aligned}$$

and

$$\begin{aligned} U &= \tfrac{1}{2} k \big(\delta X\big)^2 + \tfrac{1}{2} k \big(\delta Y\big)^2 \\ &= \tfrac{1}{2} k a^2 \left\{ \big(\sqrt{2}\, \cos\phi - 1\big)^2 + \big(\sqrt{2}\, \sin\phi - 1\big)^2 \right\} \\ &= \tfrac{1}{2} k a^2 \left\{ 3 - 2\sqrt{2}(\cos\phi + \sin\phi) \right\} \ . \end{aligned}$$

Note that minimizing $U(\phi)$ gives $\sin\phi = \cos\phi$, *i.e.* $\phi_{\rm eq} = \frac{\pi}{4}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} M a^2 \, \dot{\phi}^2 + \sqrt{2}\, k a^2 \big(\cos\phi + \sin\phi \big) + \text{const.}$$

The equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \qquad \Rightarrow \qquad Ma^2\,\ddot{\phi} = \sqrt{2}\,ka^2\,(\cos\phi - \sin\phi)$$

It's always smart to expand about equilibrium, so let's write $\phi = \frac{\pi}{4} + \delta$, which leads to

$$\ddot{\delta} + \omega_0^2\,\sin\delta = 0\ ,$$

with $\omega_0 = \sqrt{2k/M}$. This is the equation of a pendulum! Linearizing gives $\ddot{\delta} + \omega_0^2\,\delta = 0$, so the small oscillation frequency is just ω_0.

6.7 Appendix : Virial Theorem

The virial theorem is a statement about the time-averaged motion of a mechanical system. Define the *virial*,

$$G(q,p) = \sum_\sigma p_\sigma\, q_\sigma\ . \tag{6.98}$$

Then

$$\begin{aligned}\frac{dG}{dt} &= \sum_\sigma \big(\dot{p}_\sigma\, q_\sigma + p_\sigma\, \dot{q}_\sigma\big)\\ &= \sum_\sigma q_\sigma\, F_\sigma + \sum_\sigma \dot{q}_\sigma\, \frac{\partial L}{\partial \dot{q}_\sigma}\ .\end{aligned} \tag{6.99}$$

Now suppose that $T = \frac{1}{2}\sum_{\sigma,\sigma'} \mathrm{T}_{\sigma\sigma'}\,\dot{q}_\sigma \dot{q}_{\sigma'}$ is homogeneous of degree $k=2$ in $\dot{q}$, and that U is homogeneous of degree zero in $\dot{q}$. Then

$$\sum_\sigma \dot{q}_\sigma\,\frac{\partial L}{\partial \dot{q}_\sigma} = \sum_\sigma \dot{q}_\sigma\,\frac{\partial T}{\partial \dot{q}_\sigma} = 2\,T, \tag{6.100}$$

which follows from Euler's theorem on homogeneous functions.

Now consider the time average of $\dot{G}$ over a period τ:

$$\begin{aligned}\Big\langle \frac{dG}{dt}\Big\rangle &= \frac{1}{\tau}\int_0^\tau\! dt\,\frac{dG}{dt}\\ &= \frac{1}{\tau}\Big[G(\tau) - G(0)\Big]\ .\end{aligned} \tag{6.101}$$

If $G(t)$ is bounded, then in the limit $\tau \to \infty$ we must have $\langle \dot{G}\rangle = 0$. Any bounded motion, such as the orbit of the earth around the Sun, will result in $\langle \dot{G}\rangle_{\tau\to\infty} = 0$. But then

$$\Big\langle \frac{dG}{dt}\Big\rangle = 2\,\langle T\rangle + \Big\langle \sum_\sigma q_\sigma\, F_\sigma\Big\rangle = 0\ , \tag{6.102}$$

which implies

$$\begin{aligned}
\langle T \rangle &= -\tfrac{1}{2}\Big\langle \sum_\sigma q_\sigma \, F_\sigma \Big\rangle = +\Big\langle \tfrac{1}{2} \sum_\sigma q_\sigma \, \frac{\partial U}{\partial q_\sigma} \Big\rangle \\
&= \Big\langle \tfrac{1}{2} \sum_i \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_{\!i} U\big(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N\big) \Big\rangle && (6.103) \\
&= \tfrac{1}{2} k \, \langle U \rangle \ , && (6.104)
\end{aligned}$$

where the last line pertains to homogeneous potentials of degree k. Finally, since $T+U=E$ is conserved, we have

$$\langle T \rangle = \frac{k\,E}{k+2} \quad , \quad \langle U \rangle = \frac{2\,E}{k+2} \ . \qquad (6.105)$$