Chapter 3

One-Dimensional Conservative Systems

3.1 Description as a Dynamical System

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \ . \tag{3.1}$$

A system is *conservative* if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \tfrac{1}{2}m\dot{x}^2 + U(x) \ , \tag{3.2}$$

is then conserved. This may be verified explicitly:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\Big[\tfrac{1}{2}m\dot{x}^2 + U(x)\Big] \\ &= \Big[m\ddot{x} + U'(x)\Big]\,\dot{x} = 0 \ . \end{aligned} \tag{3.3}$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\bigg(E - U(x)\bigg)} \ . \tag{3.4}$$

Note that the constant E is a constant of integration. The $\pm$ sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E) \ , \tag{3.5}$$

where U^{-1} is the inverse function, are called *turning points*. When the total energy is E, the motion of the system is bounded by the turning points, and confined to the region(s)

$U(x) \leq E$. We can integrate eqn. 3.4 to obtain

$$t(x) - t(x_0) = \pm\sqrt{\frac{m}{2}} \int\limits_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}} \ . \qquad (3.6)$$

This is to be inverted to obtain the function $x(t)$. Note that there are now *two* constants of integration, E and x_0. Since

$$E = E_0 = \tfrac{1}{2} m v_0^2 + U(x_0) \ , \qquad (3.7)$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0. Thus, there are two *independent* constants of integration.

For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m} \int\limits_{x_-(E)}^{x_+(E)} \frac{dx'}{\sqrt{E - U(x')}} \ . \qquad (3.8)$$

3.1.1 Example : harmonic oscillator

In the case of the harmonic oscillator, we have $U(x) = \frac{1}{2} k x^2$, hence

$$\frac{dt}{dx} = \pm\sqrt{\frac{m}{2E - kx^2}} \ . \qquad (3.9)$$

The turning points are $x \pm (E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the motion, let us substitute

$$x = \sqrt{\frac{2E}{k}} \sin\theta \ . \qquad (3.10)$$

We then find

$$dt = \sqrt{\frac{m}{k}} \, d\theta \ , \qquad (3.11)$$

with solution

$$\theta(t) = \theta_0 + \omega t \ , \qquad (3.12)$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the complete motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}} \ \sin(\omega t + \theta_0) \ . \qquad (3.13)$$

Note the two constants of integration, E and θ_0.

3.2 One-Dimensional Mechanics as a Dynamical System

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, *viz.*

$$\frac{dx}{dt} = v \tag{3.14}$$

$$\frac{dv}{dt} = \frac{1}{m}\,F(x) \ . \tag{3.15}$$

This may be written in matrix-vector form, as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m}\,F(x) \end{pmatrix} \ . \tag{3.16}$$

This is an example of a *dynamical system*, described by the general form

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi}) \ , \tag{3.17}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ is an N-dimensional vector in *phase space*. For the model of eqn. 3.16, we evidently have $N = 2$. The object $\boldsymbol{V}(\boldsymbol{\varphi})$ is called a *vector field*. It is itself a vector, existing at every point in phase space, R^N. Each of the components of $\boldsymbol{V}(\boldsymbol{\varphi})$ is a function (in general) of *all* the components of $\boldsymbol{\varphi}$:

$$V_j = V_j(\varphi_1, \ldots, \varphi_N) \qquad (j = 1, \ldots, N) \ . \tag{3.18}$$

Solutions to the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ are called *integral curves*. Each such integral curve $\boldsymbol{\varphi}(t)$ is uniquely determined by N constants of integration, which may be taken to be the initial value $\boldsymbol{\varphi}(0)$. The collection of all integral curves is known as the *phase portrait* of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$. The phase curves are thus curves of constant energy.

3.2.1 Sketching phase curves

To plot the phase curves,

(i) Sketch the potential $U(x)$.

(ii) Below this plot, sketch $v(x;E) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$.

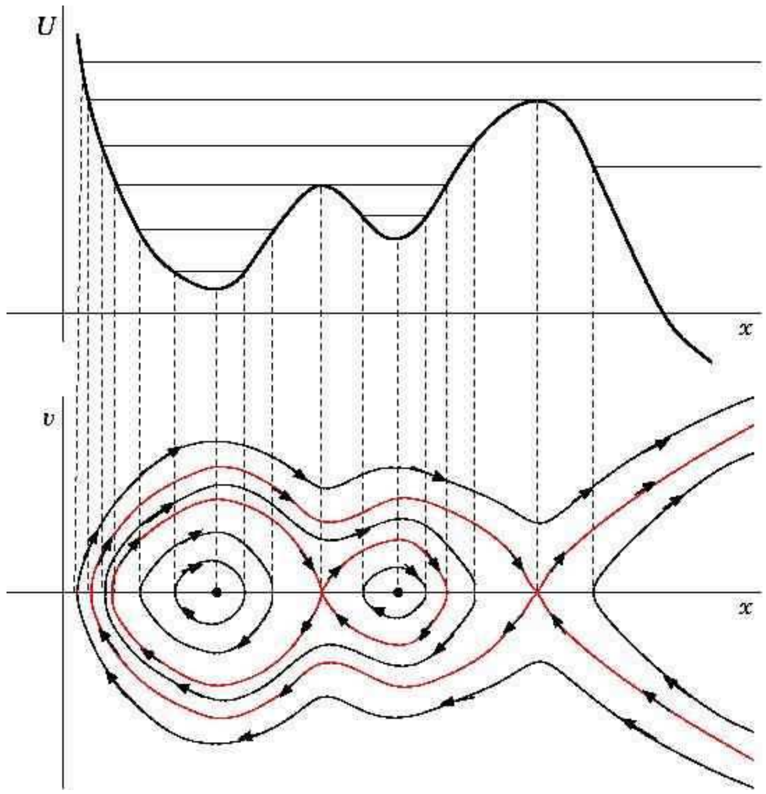

Figure 3.1: A potential $U(x)$ and the corresponding phase portraits. Separatrices are shown in red.

(iii) When E lies at a local extremum of $U(x)$, the system is at a *fixed point*.

(a) For E slightly above E_{min}, the phase curves are ellipses.

(b) For E slightly below E_{max}, the phase curves are (locally) hyperbolae.

(c) For $E = E_{\text{max}}$ the phase curve is called a *separatrix*.

(iv) When $E > U(\infty)$ or $E > U(-\infty)$, the motion is *unbounded*.

(v) Draw arrows along the phase curves: to the right for $v > 0$ and left for $v < 0$.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area $\mathcal{A}$ in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint\limits_E v\, dx = \sqrt{\frac{8}{m}} \int\limits_{x_-(E)}^{x_+(E)} dx' \sqrt{E - U(x')} \ . \tag{3.19}$$

Thus, the period is proportional to the rate of change of $A(E)$ with E:

$$T = m\, \frac{\partial \mathcal{A}}{\partial E} \ . \tag{3.20}$$

3.3 Fixed Points and their Vicinity

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^*:

$$U(x) = U(x^*) + U'(x^*)\,(x - x^*) + \tfrac{1}{2}U''(x^*)\,(x - x^*)^2 + \tfrac{1}{6}U'''(x^*)\,(x - x^*)^3 + \ldots \ . \tag{3.21}$$

Since $U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \tfrac{1}{2}k(\delta x)^2 \ , \tag{3.22}$$

where $U_0 = U(x^*)$ and $k = U''(x^*) > 0$. The solutions to the motion in this potential are:

$$U''(x^*) > 0 \ : \ \delta x(t) = \delta x_0 \, \cos(\omega t) + \frac{\delta v_0}{\omega} \, \sin(\omega t) \tag{3.23}$$

$$U''(x^*) < 0 \ : \ \delta x(t) = \delta x_0 \, \cosh(\gamma t) + \frac{\delta v_0}{\gamma} \, \sinh(\gamma t) \ , \tag{3.24}$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \tfrac{1}{2}m\,(\delta v_0)^2 + \tfrac{1}{2}k\,(\delta x_0)^2 \ . \tag{3.25}$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm\gamma\,\delta x_0$. Let's take $\delta v_0 = -\gamma\,\delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). *I.e.* the initial velocity is negative if we are to the right of the UFP ($\delta x_0 > 0$) and positive if we are to the left of the UFP ($\delta x_0 < 0$). The motion of the system is then

$$\delta x(t) = \delta x_0 \, \exp(-\gamma t) \ . \tag{3.26}$$

The particle gets closer and closer to the unstable fixed point at $\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point $\delta x < \delta x_0$ is

$$t = \gamma^{-1} \ln\left(\frac{\delta x_0}{\delta x}\right) \ . \tag{3.27}$$

This diverges logarithmically as $\delta x \to 0$. Generically, then, *the period of motion along a separatrix is infinite.*

3.3.1 Linearized dynamics in the vicinity of a fixed point

Linearizing in the vicinity of such a fixed point, we write $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{1}{m}\,U''(x^*) & 0\end{pmatrix}\begin{pmatrix}\delta x\\ \delta v\end{pmatrix} + \ldots , \tag{3.28}$$

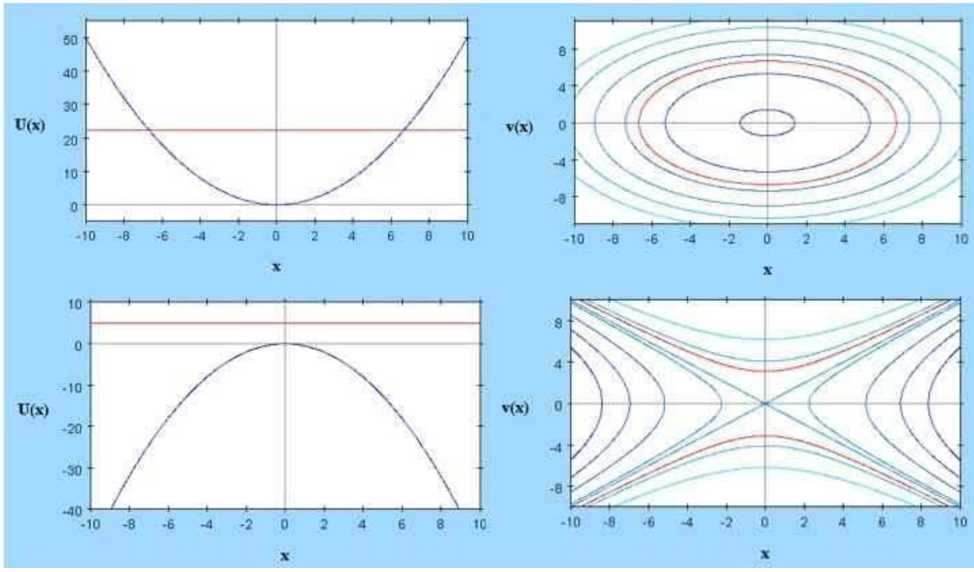

Figure 3.2: Phase curves in the vicinity of centers and saddles.

This is a *linear* equation, which we can solve completely.

Consider the general linear equation $\dot{\boldsymbol{\varphi}} = A\,\boldsymbol{\varphi}$, where A is a fixed real matrix. Now whenever we have a problem involving matrices, we should start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$A\,\boldsymbol{\psi}_\alpha = \lambda_\alpha\,\boldsymbol{\psi}_\alpha \ . \tag{3.29}$$

Here $\boldsymbol{\psi}_\alpha$ is the $\alpha^{\rm th}$ *right eigenvector*[1] of A. The eigenvalues are roots of the characteristic equation $P(\lambda) = 0$, where $P(\lambda) = \det(\lambda \cdot \mathbb{I} - A)$. Let's expand $\boldsymbol{\varphi}(t)$ in terms of the right eigenvectors of A:

$$\boldsymbol{\varphi}(t) = \sum_\alpha C_\alpha(t)\,\boldsymbol{\psi}_\alpha \ . \tag{3.30}$$

Assuming, for the purposes of this discussion, that A is nondegenerate, and its eigenvectors span R^N, the dynamical system can be written as a set of *decoupled* first order ODEs for the coefficients $C_\alpha(t)$:

$$\dot{C}_\alpha = \lambda_\alpha\,C_\alpha \ , \tag{3.31}$$

with solutions

$$C_\alpha(t) = C_\alpha(0)\,\exp(\lambda_\alpha t) \ . \tag{3.32}$$

If $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows off to infinity, while if $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows to zero. If $|\lambda_\alpha| = 1$, then $C_\alpha(t)$ oscillates with frequency $\mathsf{Im}\,(\lambda_\alpha)$.

[1] If A is symmetric, the right and left eigenvectors are the same. If A is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same.

For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D \ , \tag{3.33}$$

where $T = \mathrm{Tr}(A)$ and $D = \det(A)$. The eigenvalues are then

$$\lambda_{\pm} = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4D} \ . \tag{3.34}$$

We'll study the general case in Physics 110B. For now, we focus on our conservative mechanical system of eqn. 3.28. The trace and determinant of the above matrix are $T = 0$ and $D = \frac{1}{m}\,U''(x^*)$. Thus, there are only two (generic) possibilities: *centers*, when $U''(x^*) > 0$, and *saddles*, when $U''(x^*) < 0$. Examples of each are shown in Fig. 3.1.

3.4 Examples of Conservative One-Dimensional Systems

3.4.1 Harmonic oscillator

Recall again the harmonic oscillator, discussed in lecture 3. The potential energy is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m\,\frac{d^2x}{dt^2} = -\frac{dU}{dx} = -kx \ , \tag{3.35}$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2\, x \end{pmatrix} \ , \tag{3.36}$$

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$x(t) = x_0\,\cos(\omega t) + \frac{v_0}{\omega}\,\sin(\omega t) \tag{3.37}$$

$$v(t) = v_0\,\cos(\omega t) - \omega\, x_0\,\sin(\omega t) \ . \tag{3.38}$$

The phase curves are ellipses:

$$\omega_0\, x^2(t) + \omega_0^{-1}\, v^2(t) = C \ , \tag{3.39}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in Fig. 3.3. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \ . \tag{3.40}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{\max} = \sqrt{\frac{2E}{k}} \quad , \quad v_{\max} = \sqrt{\frac{2E}{m}} \ . \tag{3.41}$$

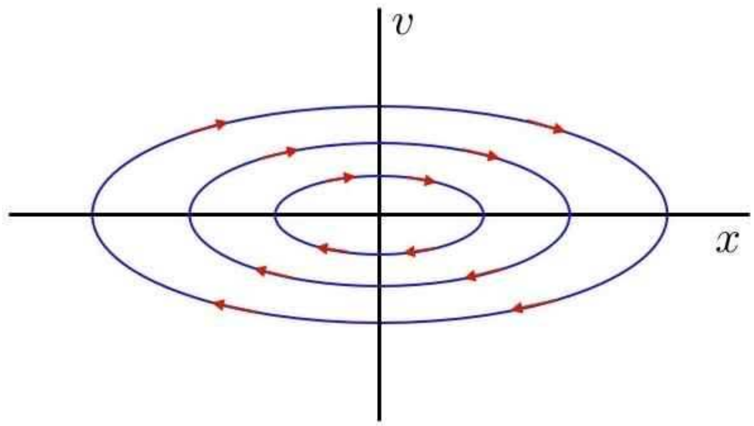

Figure 3.3: Phase curves for the harmonic oscillator.

The area of the elliptical phase curves is thus

$$\mathcal{A}(E) = \pi\, x_{\rm max}\, v_{\rm max} = \frac{2\pi E}{\sqrt{mk}} \ . \tag{3.42}$$

The period of motion is therefore

$$T(E) = m\, \frac{\partial \mathcal{A}}{\partial E} = 2\pi \sqrt{\frac{m}{k}} \ , \tag{3.43}$$

which is independent of E.

3.4.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ. The potential is $U(\theta) = -mg\ell \cos\theta$, hence

$$m\ell^2\, \ddot\theta = -\frac{dU}{d\theta} = -mg\ell \sin\theta \ . \tag{3.44}$$

This is equivalent to

$$\frac{d}{dt}\begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\omega_0^2 \sin\theta \end{pmatrix} \ , \tag{3.45}$$

where $\omega = \dot\theta$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations.

The conserved energy is

$$E = \tfrac{1}{2}\, m\ell^2\, \dot\theta^2 + U(\theta) \ . \tag{3.46}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot\theta^2 - 2\omega_0^2\, \cos\theta = -2\omega_0^2\, \cos\theta_0 \ . \tag{3.47}$$

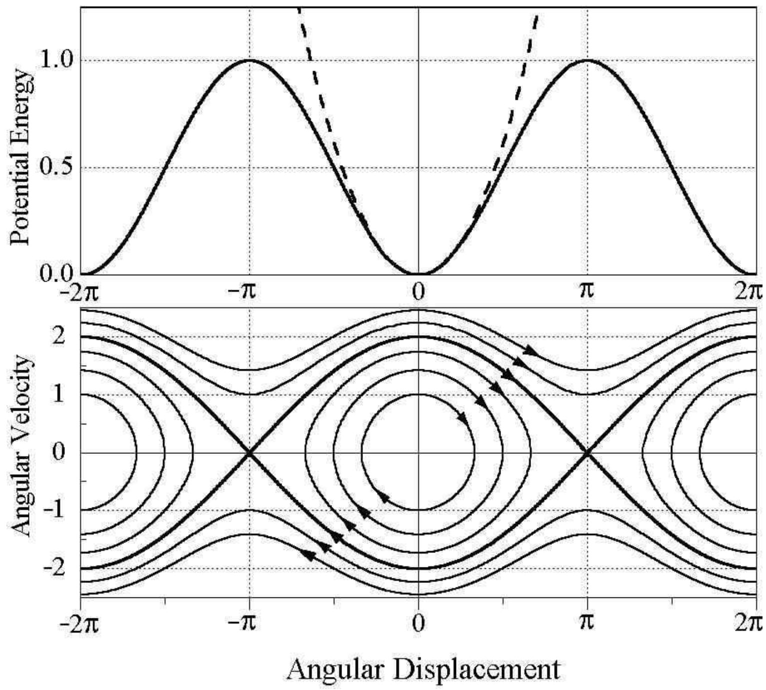

Figure 3.4: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of rotation and libration.

The period for motion of amplitude θ_0 is then

$$T(\theta_0) = \frac{\sqrt{8}}{\omega_0} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0}\,\mathrm{K}\big(\sin^2 \tfrac{1}{2}\theta_0\big) \ , \tag{3.48}$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind. Expanding $K(z)$, we have

$$T(\theta_0) = \frac{2\pi}{\omega_0}\left\{1 + \tfrac{1}{4}\sin^2\left(\tfrac{1}{2}\theta_0\right) + \tfrac{9}{64}\sin^4\left(\tfrac{1}{2}\theta_0\right) + \ldots\right\} . \tag{3.49}$$

For $\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2\,\theta$. As $\theta_0 \to \frac{\pi}{2}$, the period diverges logarithmically.

The phase curves for the pendulum are shown in Fig. 3.4. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

3.4.3 Other potentials

Using the phase plotter application written by Ben Schmidel, available on the Physics 110A course web page, it is possible to explore the phase curves for a wide variety of potentials. Three examples are shown in the following pages. The first is the effective potential for the Kepler problem,

$$U_{\rm eff}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \ , \tag{3.50}$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses $m_{1,2}$, $\mu = m_1 m_2/(m_1+m_2)$ is the 'reduced mass', and $k = G m_1 m_2$, where G is the Cavendish constant. We can then write

$$U_{\rm eff}(r) = U_0 \bigg\{ -\frac{1}{x} + \frac{1}{2x^2} \bigg\} \ , \tag{3.51}$$

where $r_0 = \ell^2/\mu k$ has the dimensions of length, and $x \equiv r/r_0$, and where $U_0 = k/r_0 = \mu k^2/\ell^2$. Thus, if distances are measured in units of r_0 and the potential in units of U_0, the potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second is the hyperbolic secant potential,

$$U(x) = -U_0 \, {\rm sech}^2(x/a) \ , \tag{3.52}$$

which, in dimensionless form, is $\mathcal{U}(x) = -{\rm sech}^2(x)$, after measuring distances in units of a and potential in units of U_0.

The final example is

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} \ . \tag{3.53}$$

Again measuring x in units of a and U in units of U_0, we arrive at $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$.

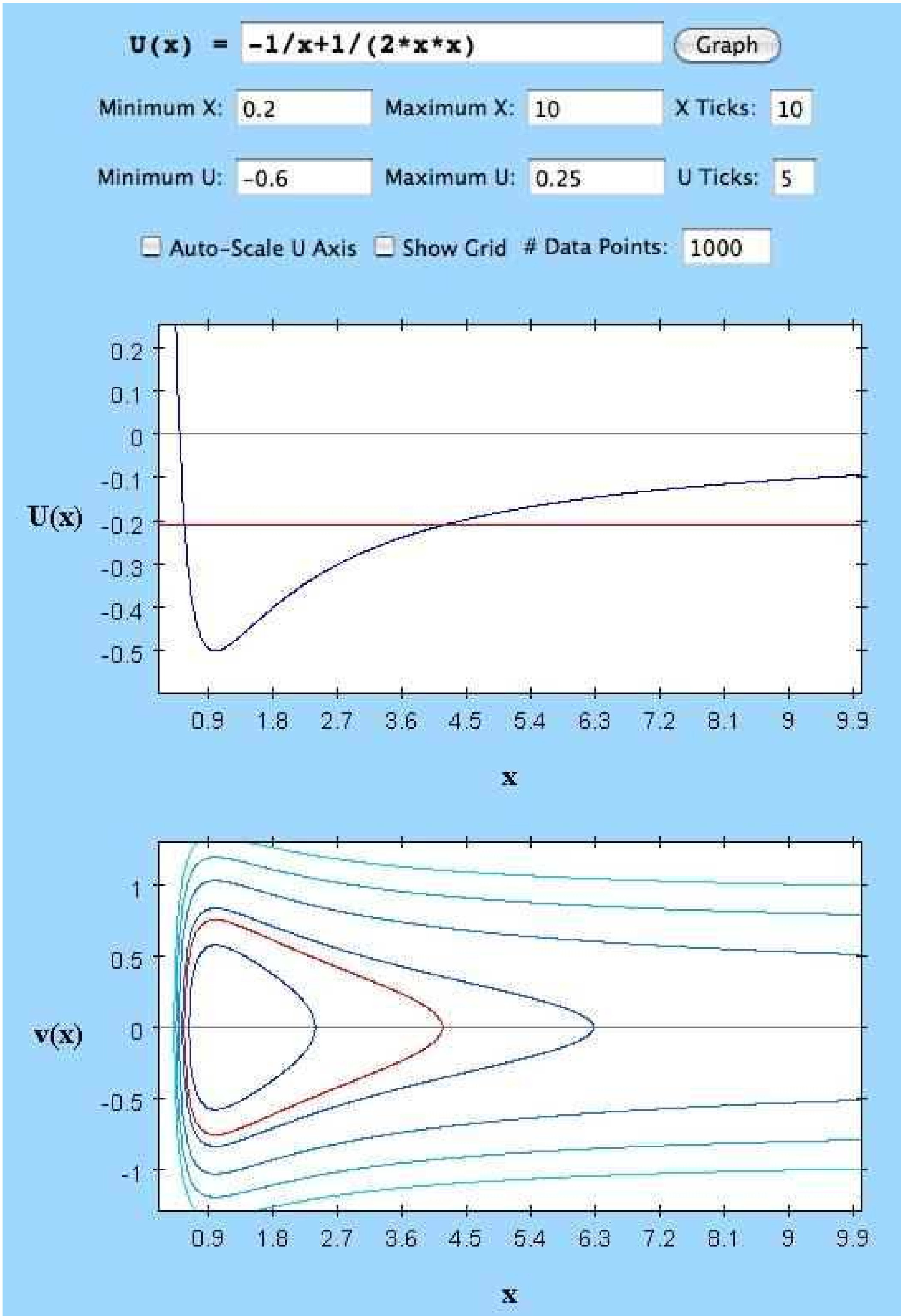

Figure 3.5: Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$.

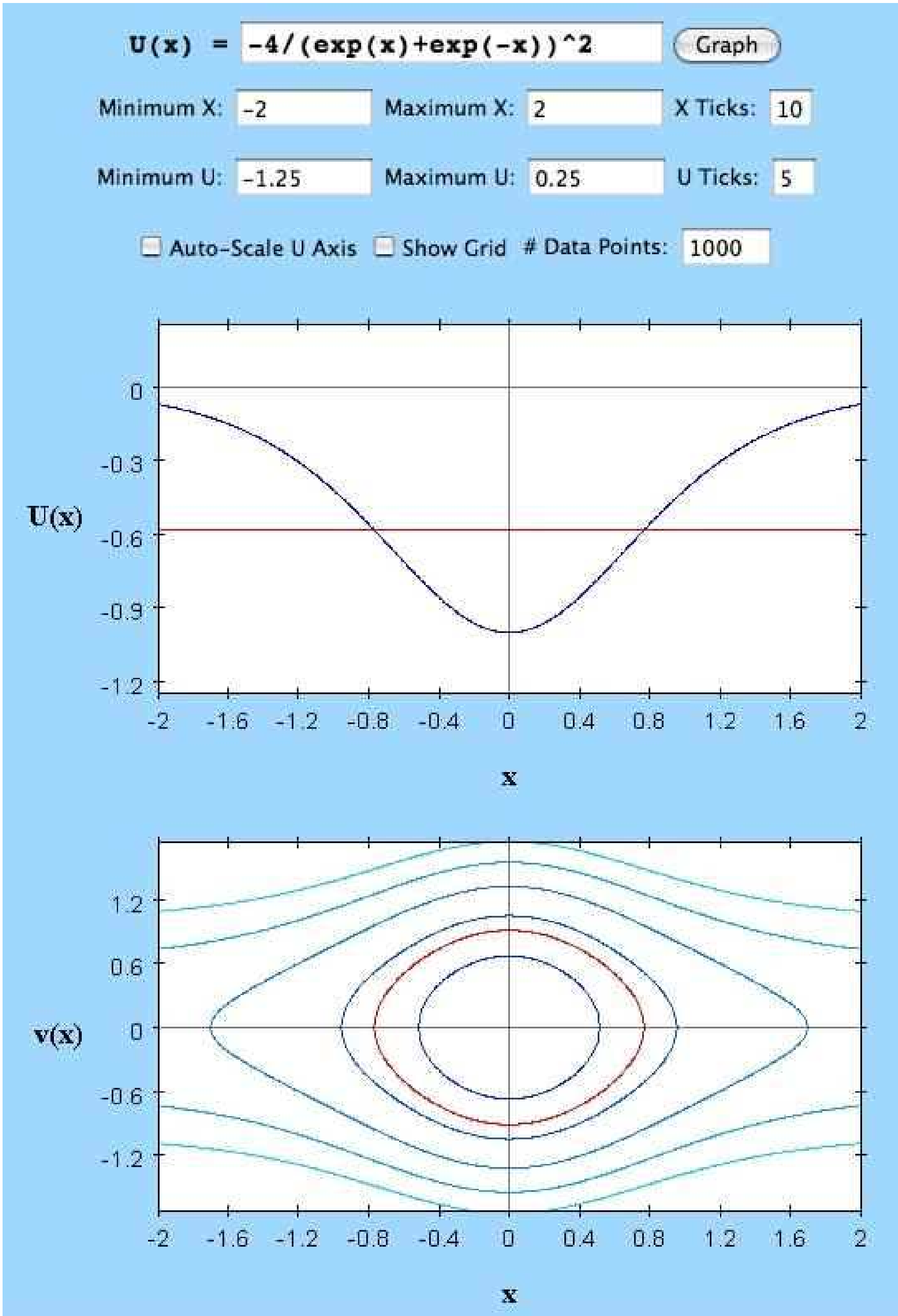

Figure 3.6: Phase curves for the potential $U(x) = -\mathrm{sech}^2(x)$.

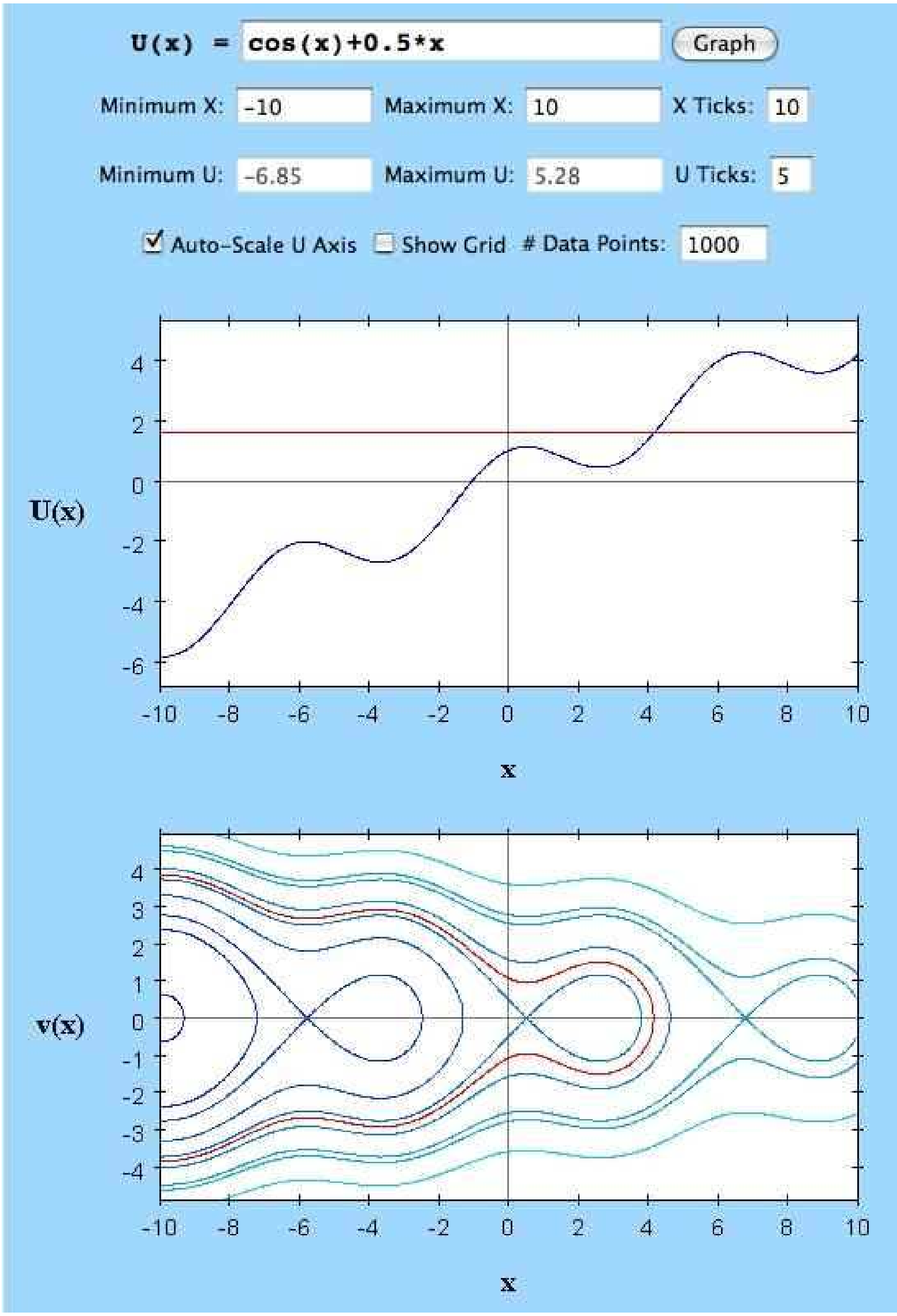

Figure 3.7: Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$.