PHYSICS 110A : CLASSICAL MECHANICS
HW 5 SOLUTIONS

(1) Taylor 7.38

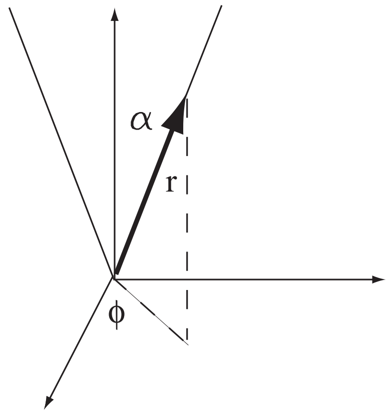

Figure 1: Figure for 7.38.

The kinetic energy will be:

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\alpha\dot{\phi}^2.$$

And the potential energy will be:

$$U = mgr\cos\alpha.$$

So our Lagrangian is:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\sin^2\alpha\dot{\phi}^2 - mgr\cos\alpha.$$

From the Euler-Lagrange equations we get:

$$\ddot{r} = r\sin^2\alpha\dot{\phi}^2 - g\cos\alpha. \tag{1}$$

And:

$$\frac{d}{dt}\left[mr^2\sin^2\alpha\dot{\phi}\right] = l_z.$$

Where l_z is a constant we know as the angular momentum in the z-direction. Solving for $\dot{\phi}$ we have $\dot{\phi} = \frac{l_z}{mr^2\sin^2\alpha}$. Let's plug this into equation (1) to get:

$$\ddot{r} = \frac{l_z^2}{m^2r^3\sin^2\alpha} - g\cos\alpha. \tag{2}$$

To find the equilibrium position we set $\ddot{r} = 0$ in equation (2) above. Therefore:

$$r_0 = \sqrt[3]{\frac{l^2}{m^2 g \sin^2\alpha \cos\alpha}}. \tag{3}$$

Finally we want to expand for small oscillations $r = r_0 + \epsilon$. So we have:

$$\ddot{\epsilon} = \frac{l_z^2}{m^2(r_0+\epsilon)^3 \sin^2\alpha} - g\cos\alpha.$$

Or:

$$\ddot{\epsilon} = \frac{l_z^2}{m^2 r_0^3 (1+\frac{\epsilon}{r_0})^3 \sin^2\alpha} - g\cos\alpha.$$

Or:

$$\ddot{\epsilon} = \frac{l_z^2}{m^2 r_0^3 \sin^2\alpha}(1 - 3\frac{\epsilon}{r_0} + ...) - g\cos\alpha.$$

But due to equation (3) we have:

$$\ddot{\epsilon} = -\frac{3 l_z^2}{m^2 r_0^4 \sin^2\alpha}\epsilon.$$

Where we have the equation for simple harmonic motion with $\omega = \frac{\sqrt{3} l_z}{m r_0^2 \sin\alpha}$.

(2) Taylor 7.39

For the Lagrangian we get:

$$L = \frac{1}{2} m \left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2\right] - U(r).$$

Which lead to the equations of motion:

$$m\ddot{r} = -\frac{dU(r)}{dr} + (mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2), \tag{4}$$

and,

$$\frac{d}{dt}[mr^2 \sin^2\theta\dot{\phi}] = 0, \tag{5}$$

and,

$$\frac{d}{dt}[mr^2\dot{\theta}] = 2mr^2 \sin\theta\cos\theta\dot{\phi}^2. \tag{6}$$

Equation (4) is Newton's second law with the force from potential term $-\frac{dU(r)}{dr}$ as well as a centrifugal force term $mr\dot{\theta}^2 + mr\sin^2\theta\dot{\phi}^2$.

Equation (5) shows that the l_z is conserved.

Equation (6) shows that the l_ϕ is conserved, however since the $\hat{\phi}$ vector is constantly changing the right hand side is not zero.

For $\theta_0 = \pi/2$ and $\dot{\theta}_0 = 0$ we have from equation (6):

$$\frac{d}{dt}[mr^2\dot{\theta}] = 0.$$

Or,

$$mr^2\dot{\theta} = C.$$

So θ remains $\pi/2$ and the object will remain in that plane.

For $\phi_0 = \phi_0$ and $\dot{\phi}_0 = 0$ we have from equation (5):

$$mr^2\sin^2\theta\dot{\phi} = C,$$

So ϕ remains ϕ_0 and the object will remain in that vertical plane.

(3) Taylor 7.41

Our parabola has the shape:

$$z = k\rho^2.$$

Which gives us a relationship between $\dot{\rho}$ and $\dot{z}$:

$$\dot{z} = 2k\rho\dot{\rho}.$$

For the Lagrangian we get:

$$L = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 + \frac{1}{2}m\dot{z}^2 - mgz.$$

Which we can plug the above constraints to get:

$$L = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 + 2mk^2\rho^2\dot{\rho}^2 - mgk\rho^2.$$

Which cleans up to look like:

$$L = \frac{1}{2}m(1 + 4k^2\rho^2)\dot{\rho}^2 + \frac{1}{2}m(\omega^2 - 2gk)\rho^2.$$

Finding the equation of motion we get:

$$\frac{d}{dt}\left[m(1 + 4k^2\rho^2)\dot{\rho}\right] = m[\omega^2 - 2gk]\rho + 4mk^2\rho\dot{\rho}^2.$$

Or:

$$(1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 = [\omega^2 - 2gk]\rho. \tag{7}$$

Assuming $\dot{\rho}_0 = 0$ equilibrium will occur when the right hand side is zero, so for $\rho = 0$ and $\omega^2 = 2gk$.

Now for small ρ and $\dot{\rho}$ we can rewrite equation (7) as:

$$\ddot{\rho} \approx [\omega^2 - 2gk]\rho.$$

So this force is similar to a spring force of the shape $F = kx$. Now when $2gk > \omega^2$ the k constant is negative and it is a restoring force. For $2gk < \omega^2$ the k constant is positive and it is not a restoring force

(4) **Taylor 7.50**

For the Lagrangian we get:

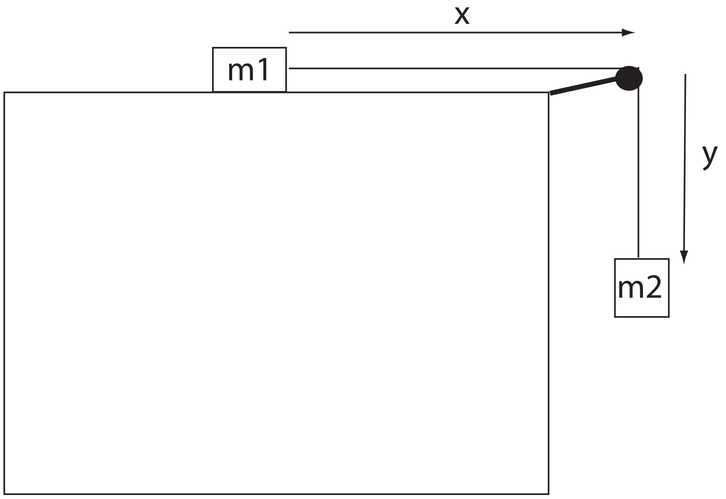

Figure 2: Figure for 7.50.

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_2gy.$$

And our equation of constraint is:

$$f = x + y - l.$$

From this our Lagrange multiplier equation leads us to:

$$m_1\ddot{x} = \lambda, \tag{8}$$

and

$$m_2\ddot{y} - m_2g = \lambda. \tag{9}$$

From our constraint equation we get:

$$\ddot{x} = -\ddot{y}.$$

Solving for λ we get,

$$\lambda = \frac{-m_1 m_2 g}{m_1 + m_2}.$$

If we were to look at this with Newton's second law we would get two equations:

$$-T = m_1 a_x,$$

and,

$$m_2 g - T = m_2 a.$$

Comparing with equations above we see $\lambda = -T$.

(5) Taylor 7.51

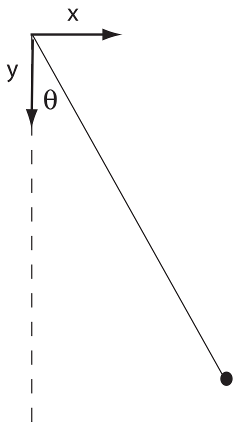

Figure 3: Figure for 7.51.

$$L = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 + mgy.$$

And our equation of constraint is:

$$f = \sqrt{x^2 + y^2} - l.$$

From this our Lagrange multiplier equation leads us to:

$$m\ddot{x} = \lambda \frac{x}{\sqrt{x^2 + y^2}}, \tag{10}$$

and

$$m\ddot{y} - mg = \lambda \frac{y}{\sqrt{x^2 + y^2}}. \tag{11}$$

Now calling θ the angle from the vertical we can rewrite these as:

$$m\ddot{x} = \lambda sin\theta, \tag{12}$$

and

$$m\ddot{y} - mg = \lambda cos\theta. \tag{13}$$

Writing out equations from Newton's second law we get:

$$m\ddot{x} = -Tsin\theta,$$

and

$$m\ddot{y} = -Tcos\theta + mg.$$

So we see $\lambda = -T$.

If we were to use the constraint equation:

$$f = x^2 + y^2 - l^2.$$

We get for our equations of motion:

$$m\ddot{x} = \lambda 2x,$$

and

$$m\ddot{y} - mg = \lambda 2y.$$

Getting rid of λ in the equations (12) and (13) we get:

$$\frac{m\ddot{x}y}{x} = m\ddot{y} - mg,$$

Which is exactly what you get getting rid of lambda in equations (10) and (11).