**PHYSICS 110A : CLASSICAL MECHANICS**
**HW 4 SOLUTIONS**

## (2) Taylor 7.14

For the yo-yo the kinetic energy will have a rotational and translational motion:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

Now in our coordinate system $v = \dot{x}$ and $\omega = \dot{\phi}$. We also know the moment of inertia for a solid disk is $I = \frac{1}{2}mR^2$. Finally since the rope does not slip as the yo-yo falls we can say $v = \omega R$ (remember this equation?), or for us $\dot{x} = \dot{\phi}R$. Altogether we have:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{\dot{x}}{R}\right)^2 = \frac{3}{4}m\dot{x}^2.$$

Our potential energy is:

$$U = -mgx.$$

So out Lagrangian is:

$$L = \frac{3}{4}m\dot{x}^2 + mgx.$$

using the Euler-Lagrange equation we find:

$$\ddot{x} = \frac{2}{3}g.$$

## (3) Taylor 7.27

Here the kinetic energy is (please see figure 1):

$$T = \frac{1}{2}4m\dot{x}^2 + \frac{1}{2}m(-\dot{x} + \dot{y})^2 + +\frac{1}{2}3m(-\dot{x} + -\dot{y})^2 = 4m\dot{x}^2 + 2m\dot{y}^2 + 2m\dot{x}\dot{y}.$$

And our potential will be:

$$U = -4mgx - mg(l_1 - x + y) - 3mg(l_1 - x + l_2 - y) = mg2y.$$

Where we dropped any constants. So our Lagrangian is:

$$L = 4m\dot{x}^2 + 2m\dot{y}^2 + 2m\dot{x}\dot{y} - mg2y.$$

After we Euler-Lagrange it we have:

$$4\ddot{x} = -\ddot{y}.$$

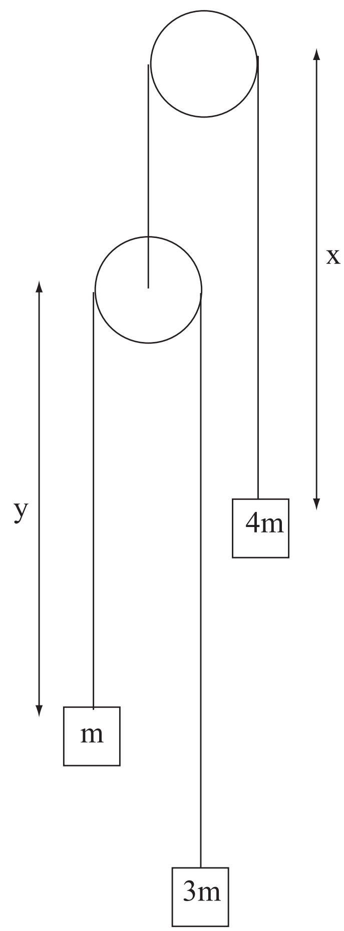


Figure 1: Figure for 7.27.

And:

$$2\ddot{y} + \ddot{x} = -g.$$

You can solve these coupled equations to get:

$$\ddot{x} = \frac{g}{7}.$$

## (4) **Taylor 7.33**

We find the kinetic energy for the bar of soap (useful for making money to sustain your fight club). There is both a rotational and translational term (please see figure 2).

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2.$$

Where $x$ is the distance from the soap to the edge of the plate. And our potential will be:

$$U = mgx\sin(\omega t).$$

So our Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2 - mgx\sin(\omega t).$$

Remember $\omega$ is a constant.

After we Euler-Lagrange it we have:

$$-mg\sin(\omega t) + mx\omega^2 - m\ddot{x} = 0.$$

Or:

$$\ddot{x} - \omega^2 x = -g\sin(\omega t). \tag{1}$$

For the solution it is suggested to try a technique similar to what we used to solve equation 5.48 in the text. Here we try a solution of the form:

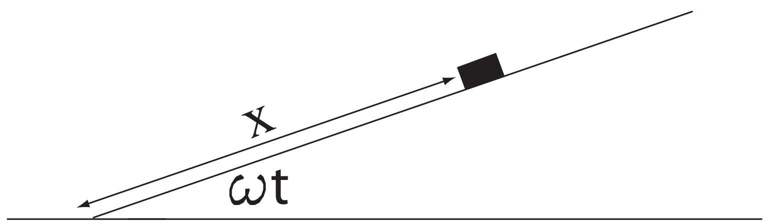


Figure 2: Plot keeping 4 terms.

$$x(t) = x_p(t) + x_h(t).$$

The homogeneous equation is:

$$\ddot{x} - \omega^2 x = 0.$$

The solution to this is:

$$x_h(t) = c_1 e^{(\omega t)} + c_2 e^{(-\omega t)}.$$

Or similarly:

$$x_h(t) = r_1\cosh(\omega t) + r_2\sinh(\omega t).$$

And for the particular solution they suggest trying:

$$x_p(t) = A\sin(\omega t).$$

Plugging this into equation (1) we get:

$$-\omega^2 A\sin(\omega t) - \omega^2 A\sin(\omega t) = -g\sin(\omega t).$$

Solving for $A$ we have:

$$A = \frac{g}{2\omega^2}.$$

So our solution looks like:

$$x(t) = r_1 \cosh(\omega t) + r_2 \sinh(\omega t) + \frac{g}{2\omega^2} \sin(\omega t). \tag{2}$$

We have initial conditions:

$$x(0) = x_0,$$

and

$$\dot{x}(0) = 0.$$

These combined with equation (2) lead us the the solution for $x(t)$:

$$x(t) = x_0 \cosh(\omega t) + \frac{g}{2\omega^2} \left(\sin(\omega t) - \sinh(\omega t)\right). \tag{3}$$

## (5) Taylor 7.36

**(a)** The kinetic energy will be:

$$T = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2;$$

and the potential energy will be:

$$U = \frac{1}{2} k(l_0 - r)^2 - mgr\cos\phi.$$

So our Lagrangian is:

$$L = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 - \frac{1}{2} k(l_0 - r)^2 + mgr\cos\phi.$$

**(b)** From the Euler-Lagrange equations we get:

$$\ddot{r} = r\dot{\phi}^2 + g\cos\phi + \frac{k}{m}(l_0 - r);$$

and:

$$\frac{d}{dt}\left[r^2\dot{\phi}\right] = -gr\sin\phi.$$

**(c)** Now we are asked to solve these equations for small oscillations. For small oscillations these equations become:

$$\ddot{r} = g + \frac{k}{m}(l_0 - r); \tag{4}$$

And:

$$\frac{d}{dt}\left[r^2\dot{\phi}\right] = -gr\phi. \tag{5}$$

Setting equation (4) equal to zero we find the equilibrium value or $r$ is $r_0 = \frac{m}{k} g + l_0$.

Now we will expand for small values of $r$ by setting $r = r_0 + \epsilon$.
We have:

$$(r_0 \ddot{+} \epsilon) = g + \frac{k}{m}(l_0 - r_0 - \epsilon);$$

Or:

$$\ddot{\epsilon} = g + \frac{k}{m}(l_0 - \frac{m}{k}g - l_0 - \epsilon).$$

And finally:

$$\ddot{\epsilon} = -\frac{k}{m}\epsilon;$$

Where we have the equation for simple harmonic motion with $\omega^2 = \frac{k}{m}$. And for equation (5) expanding out we have:

$$\ddot{\phi} = -\frac{g}{r_0}\phi. \tag{6}$$

Where we have the equation for simple harmonic motion with $\omega^2 = \frac{g}{r_0}$.

### (6) Taylor 7.37

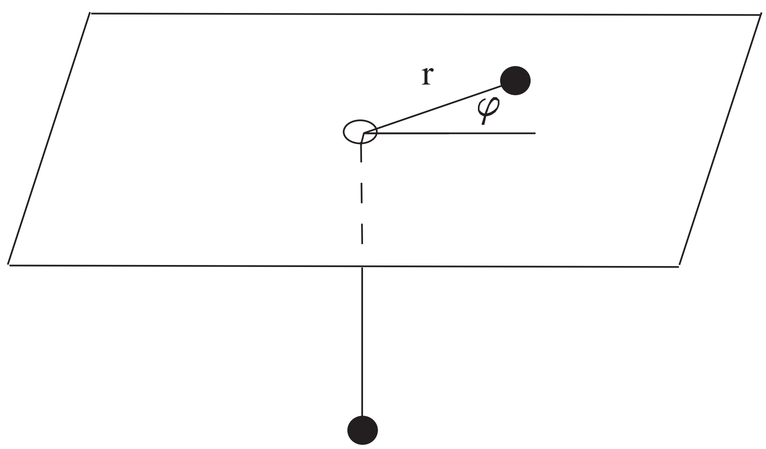


Figure 3: Figure for 7.37.

**(a)** The kinetic energy will be:

$$T = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2.$$

Note: here we have two blocks that will have translational kinetic energy. And the potential energy will be:

$$U = mgr.$$

So our Lagrangian is:

$$L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr.$$

**(b)** From the Euler-Lagrange equations we get:

$$2\ddot{r} = r\dot{\phi}^2 - g. \tag{7}$$

And:

$$\frac{d}{dt}\left[mr^2\dot{\phi}\right] = l.$$

Where $l$ is a constant we know as the angular momentum. So in this problem the angular momentum is conserved. Solving for $\dot{\phi}$ we have $\dot{\phi} = \frac{l}{mr^2}$. Let's plug this into equation (7) to get:

$$2\ddot{r} = \frac{l^2}{m^2r^3} - g. \tag{8}$$

**(c)** To find the equilibrium position we set $\ddot{r} = 0$ in equation (8) above. Therefore:

$$r_0 = \sqrt[3]{\frac{l^2}{m^2g}}. \tag{9}$$

Physically if you plug the definition of $l$ back in to equation (9) you get:

$$g = r\dot{\phi}^2.$$

Which is to say the gravitational acceleration of the falling block is equal to the centripetal acceleration of the rotating block.

**(d)** Finally we want to expand for small oscillations $r = r_0 + \epsilon$. So we have:

$$2\ddot{\epsilon} = \frac{l^2}{m^2(r_0+\epsilon)^3} - g.$$

Or:

$$2\ddot{\epsilon} = \frac{l^2}{m^2r_0^3(1+\frac{\epsilon}{r_0})^3} - g.$$

Or:

$$2\ddot{\epsilon} = \frac{l^2}{m^2r_0^3}(1 - 3\frac{\epsilon}{r_0} + ...) - g.$$

But due to equation (9) we have:

$$\ddot{\epsilon} = -\frac{3l^2}{2m^2r_0^4}\epsilon.$$

Where we have the equation for simple harmonic motion with $\omega^2 = \sqrt{\frac{3}{2}}\frac{l}{mr_0^2}$.