PHYSICS 110A : CLASSICAL MECHANICS
MIDTERM EXAM #2

A mechanical system consists of a ring of radius a and mass M, and a point particle of mass m configured as shown in the sketch below. The ring is affixed to a massless rigid rod of length ℓ which is free to swing in a plane (the plane of the ring). The point mass m moves along the inner surface of the ring. The apparatus moves under gravity.

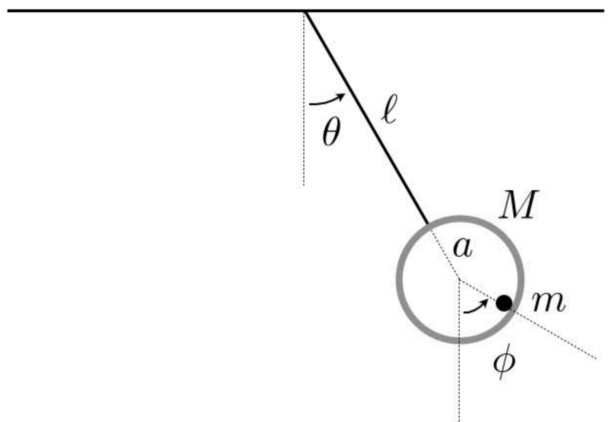

Figure 1: A point mass m slides frictionlessly inside a ring of radius a and mass M which is affixed to a rigid rod of length ℓ. The apparatus moves under the influence of gravity.

(a) Choose as generalized coordinates the angles θ and ϕ shown in the figure. Express the Cartesian coordinates (x, y) of the point mass in terms of the angles θ and ϕ and the lengths ℓ and a. Note that the center of the ring lies a distance $(\ell + a)$ from the fulcrum.
[20 points]

Solution: Clearly

$$\begin{aligned} x &= (\ell + a)\sin\theta + a\sin\phi \\ y &= -(\ell + a)\cos\theta - a\cos\phi \ . \end{aligned}$$

(b) Find the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi}, t)$. You may find it convenient to abbreviate $\ell + a \equiv b$.
[20 points]

Solution: The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2} M(\ell + a)^2 \dot{\theta}^2 + \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) \\ &= \tfrac{1}{2}(M + m)b^2\,\dot{\theta}^2 + mab\cos(\theta - \phi)\,\dot{\theta}\,\dot{\phi} + \tfrac{1}{2} ma^2 \dot{\phi}^2 \ . \end{aligned}$$

The potential energy is

$$\begin{aligned} U &= -Mg(\ell + a)\cos\theta + mgy \\ &= -(M + m)gb\cos\theta - mga\cos\phi \ . \end{aligned}$$

Thus, the Lagrangian $L = T - U$ is

$$L = \tfrac{1}{2}(M+m)\, b^2\, \dot\theta^2 + mab\cos(\theta-\phi)\, \dot\theta\, \dot\phi + \tfrac{1}{2} ma^2\, \dot\phi^2 + (M+m)gb\cos\theta + mga\cos\phi \ .$$

(c) Find p_θ, p_ϕ, F_θ, and F_ϕ.
`[20 points]`

Solution: We have

$$\begin{aligned}
p_\theta &= \frac{\partial L}{\partial\dot\theta} = (M+m)\, b^2\, \dot\theta + mab\cos(\theta-\phi)\, \dot\phi \\
p_\phi &= \frac{\partial L}{\partial\dot\phi} = ma^2\, \dot\phi + mab\cos(\theta-\phi)\, \dot\theta \\
F_\theta &= \frac{\partial L}{\partial\theta} = -mab\sin(\theta-\phi)\, \dot\theta\, \dot\phi - (M+m)gb\sin\theta \\
F_\phi &= \frac{\partial L}{\partial\phi} = mab\sin(\theta-\phi)\, \dot\theta\, \dot\phi - mga\sin\phi \ .
\end{aligned}$$

(d) Write down the equations of motion in terms of the generalized coordinates and their first and second time derivatives.
`[20 points]`

Solution: We have $\dot p_\theta = F_\theta$ and $\dot p_\phi = F_\phi$. Note that $\frac{d}{dt}\cos(\theta-\phi) = (\dot\phi - \dot\theta)\sin(\theta-\phi)$. Thus,

$$\begin{aligned}
(M+m)\, b^2\, \ddot\theta + mab\cos(\theta-\phi)\ddot\phi + mab\, \sin(\theta-\phi)\, \dot\phi^2 &= -(M+m)gb\, \sin\theta \\
mab\cos(\theta-\phi)\ddot\theta - mab\sin(\theta-\phi)\, \dot\theta^2 + ma^2\, \ddot\phi &= -mga\sin\phi \ .
\end{aligned}$$

(e) What, if anything is conserved? Express all conserved quantities in terms of the generalized coordinates and velocities.
`[20 points]`

Solution: Since $\frac{\partial L}{\partial t} = 0$, we have that $H = p_\theta\, \dot\theta + p_\phi\, \dot\phi - L$ is conserved. This is the only conserved quantity. Since the kinetic energy T is homogeneous of degree $k=2$ in the generalized velocities, we have $H = E = T + U$. Thus,

$$E = \tfrac{1}{2}(M+m)\, b^2\, \dot\theta^2 + mab\cos(\theta-\phi)\, \dot\theta\, \dot\phi + \tfrac{1}{2} ma^2\, \dot\phi^2 - (M+m)gb\cos\theta - mga\cos\phi \ .$$

(f) Find the tension in the massless rod in terms of θ, ϕ, and their first and second time derivatives.
`[20 quatloos extra credit]`

Solution: Now we write

$$\begin{aligned}
x &= (r+a)\sin\theta + a\sin\phi \\
y &= -(r+a)\cos\theta - a\cos\phi
\end{aligned}$$

with the constraint $G = r - \ell = 0$. Then

$$\begin{aligned}\dot{x} &= (r+a)\cos\theta\,\dot\theta + a\cos\phi\,\dot\phi + \dot r\,\sin\theta \\ \dot{y} &= (r+a)\sin\theta\,\dot\theta + a\sin\phi\,\dot\phi - \dot r\cos\theta\ .\end{aligned}$$

The Lagrangian becomes

$$\begin{aligned}L = \tfrac{1}{2}(M+m)\,(r+a)^2\,\dot\theta^2 + \tfrac{1}{2}(M+m)\,\dot r^2 + \tfrac{1}{2}ma^2\,\dot\phi^2 + ma(r+a)\cos(\theta-\phi)\,\dot\theta\,\dot\phi \\ + ma\sin(\theta-\phi)\,\dot r\,\dot\phi + (M+m)\,g\,(r+a)\cos\theta + mga\cos\phi\ .\end{aligned}$$

We now have

$$\begin{aligned}p_r &= \frac{\partial L}{\partial \dot r} = (M+m)\,\dot r + ma\sin(\theta-\phi)\,\dot\phi \\ F_r &= \frac{\partial L}{\partial r} = (M+m)\,(r+a)\,\dot\theta^2 + ma\cos(\theta-\phi)\,\dot\theta\,\dot\phi + (M+m)\,g\,\cos\theta \\ Q_r &= \lambda\,\frac{\partial G}{\partial r} = \lambda\ .\end{aligned}$$

The equation of motion for r is

$$\dot p_r = F_r + Q_r\ ,$$

hence

$$\begin{aligned}Q_r &= \dot p_r - F_r \\ &= ma\sin(\theta-\phi)\,\ddot\phi + ma\cos(\theta-\phi)\,(\dot\theta-\dot\phi)\,\dot\phi \\ &\qquad - (M+m)\,b\,\dot\theta^2 - ma\cos(\theta-\phi)\,\dot\theta\,\dot\phi - (M+m)\,g\,\cos\theta\ ,\end{aligned}$$

where we have invoked $r = \ell$. Thus,

$$Q_r = ma\sin(\theta-\phi)\,\ddot\phi - ma\cos(\theta-\phi)\,\dot\phi^2 - (M+m)\,b\,\dot\theta^2 - (M+m)\,g\,\cos\theta$$

and the tension is $\mathsf{T} = -Q_r$.