PHYSICS 110A : CLASSICAL MECHANICS
FALL 2010 FINAL EXAMINATION

(1) A point mass m_1 slides frictionlessly along a curve $y = f(x)$, as depicted in Fig. 1. Affixed to the mass is a rigid rod of length ℓ, at the other end of which is a second point mass m_2. The entire apparatus moves under the influence of gravity. Choose as generalized coordinates the set $\{x, y, \theta\}$, where (x, y) are the Cartesian coordinates of the mass m_1, and θ is the angle shown in the figure. Treat the condition $y = f(x)$ as a constraint.

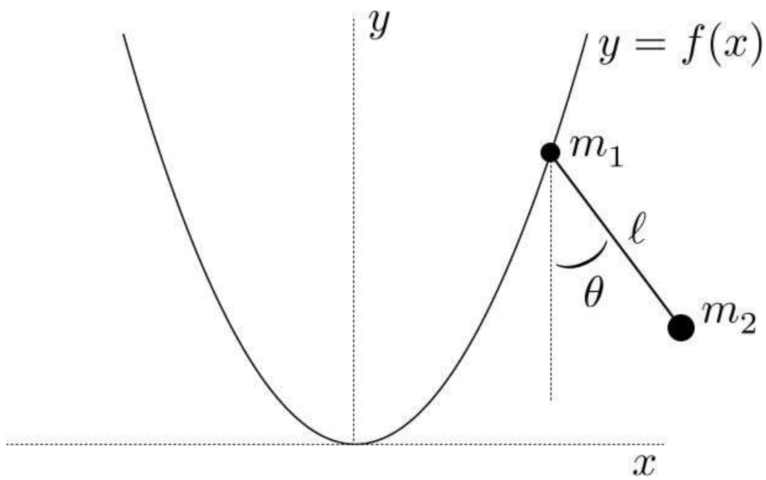

Figure 1: A mass point m_1 moves frictionlessly along the curve $y = f(x)$. Affixed to this mass is a rigid rod of length ℓ at the end of which is a second point mass m_2.

(a) Find the Lagrangian $L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, t)$. `[5 points]`

(b) Find the momenta p_x, p_y, and p_θ. `[6 points]`

(c) Find the forces F_x, F_y, and F_θ. `[3 points]`

(d) Find the forces of constraint Q_x, Q_y, and Q_θ. `[3 points]`

(e) Find the equations of motion in terms of x, y, θ, their first and second time derivatives, and the Lagrange multiplier λ. `[6 points]`

(f) What is conserved for this system? `[2 points]`

(2) Treat the system described in problem (1) without the constraint formalism, using generalized coordinates x and θ. Assume $f(x) = f(-x)$ is a symmetric function with a single minimum at $x = 0$, and that $f''(0) > 0$.

(a) Find the Lagrangian $L(x, \theta, \dot{x}, \dot{\theta}, t)$. `[5 points]`

(b) Find the equilibrium values (x^*, θ^*) and the T and V matrices. `[5 points]`

(c) Consider the case $f(x) = x^2/2b$. Define $\Omega_0 = \sqrt{g/b}$ and $\Omega_1 = \sqrt{g/\ell}$. Find a general expression for the normal mode frequencies $\omega_\pm$. Then consider the case where $m_1 = 21m$, $m_2 = 4m$, $\Omega_0 = 3\Omega$, and $\Omega_1 = 5\Omega$. Find $\omega_\pm$. `[10 points]`

(d) Find the eigenvectors $\boldsymbol{\psi}^{(\pm)}$. *You do not have to normalize them.* `[5 points]`

(3) Two particles of masses m_1 and m_2 interact via the central potential

$$U(\boldsymbol{r}_1, \boldsymbol{r}_2) = -U_0 \left(\frac{a}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \right)^{1/2} .$$

(a) Find and sketch the effective potential $U_{\text{eff}}(r)$. Sketch the phase curves in the $(r, \dot{r})$ plane. Identify any separatrices and find their energies. `[5 points]`

(b) Find the radius r_0 of the circular orbit as a function of the angular momentum ℓ and other constants. `[5 points]`

(c) Writing $r(t) = r_0 + \eta(t)$, find the linearized equations of motion for $\eta(t)$. Find the frequency ω of the radial oscillations. `[5 points]`

(d) What is the shape of the nearly circular orbits? What is the shape $r(\phi)$ of the nearly circular orbits? Are those orbits closed? Why or why not? `[5 points]`

(e) What is the ratio of the escape velocity at r_0 to the orbital velocity at r_0? `[5 points]`

(4) Provide brief but substantial answers to the following questions.

(a) For a system with kinetic and potential energies

$$T = \tfrac{1}{2} m\big(\dot{x}^2 + \omega^2 x^2\big) \qquad , \qquad U = U(x) \ ,$$

find the Hamiltonian. Under what conditions is $H = T + U$? `[5 points]`

(b) For central force motion, what is the definition of a bounded orbit? What is a closed orbit? What are the conditions for a circular orbit, and under what conditions is a circular orbit stable with respect to small perturbations? Under what conditions is an almost circular orbit closed? `[5 points]`

(c) Write down an example of a Lagrangian for a system with two generalized coordinates (and no constraints), and which yields two and only two conserved quantities. `[5 points]`

(d) Consider the functional

$$F[y(x)] = \int\limits_{-\infty}^{\infty} dx \left[\tfrac{1}{2} a \left(\frac{d^2y}{dx^2} \right)^2 + \tfrac{1}{2} b \left(\frac{dy}{dx} \right)^2 + \tfrac{1}{2} c y^2 - j(x)\, y(x) \right]$$

What is the differential equation which extremizes $F[y(x)]$? `[5 points]`

(e) Consider an equilateral triangle composed of three point masses connected by three springs which moves in a horizontal plane. How many normal modes of oscillation are there? Some of the normal modes involve no restoring force. Can you identify the type of motion for these three 'zero modes'? `[5 points]`