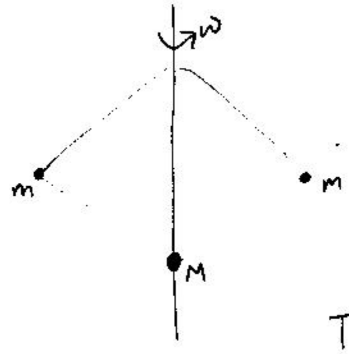
1.



a) $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2)$

$\dot{\phi} = \omega$, for mass m; $r = l$, $\dot{r} = 0$

for mass M; $r = 2l\cos\theta$, $\dot{r} = -2l\sin\theta\,\dot{\theta}$

$$T = m(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\omega^2) + \frac{1}{2}M(4l^2\sin^2\theta\,\dot{\theta}^2$$

$$U = -2mgl\cos\theta - 2Mgl\cos\theta = -4(m+M)gl\cos\theta$$

$$L = ml^2(\dot{\theta}^2 + \sin^2\theta\,\omega^2) + 2Ml^2\sin^2\theta\,\dot{\theta}^2 - 4(m+M)gl\cos\theta$$

b) $\frac{\partial L}{\partial \theta} = 2ml^2\sin\theta\cos\theta\,\omega^2 + 4Ml^2\sin\theta\cos\theta\,\dot{\theta}^2 + 4(m+M)gl\sin\theta$

$$\frac{\partial L}{\partial \dot{\theta}} = 2ml^2\dot{\theta} + 4Ml^2\sin^2\theta\,\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2ml^2\ddot{\theta} + 4Ml^2\sin^2\theta\,\ddot{\theta} + 8Ml^2\sin\theta\cos\theta\,\dot{\theta}^2$$

$$2ml^2\sin\theta\cos\theta\,\omega^2 - 4Ml^2\sin\theta\cos\theta\,\dot{\theta}^2 + 4(m+M)gl\sin\theta = 2ml^2\ddot{\theta} + 4Ml^2\sin^2\theta\,\ddot{\theta}$$

c) $H = ml^2\dot{\theta}^2 + 2Ml^2\sin^2\theta\,\dot{\theta}^2 - ml^2\sin^2\theta\,\omega^2 - 4(m+M)gl\cos\theta$

$$U_{eff} = -4(m+M)gl\cos\theta - ml^2\sin^2\theta\,\omega^2$$

$$\frac{\partial U_{eff}}{\partial \theta} = 4(m+M)gl\sin\theta - 2ml^2\sin\theta\cos\theta\,\omega^2 = 0$$

$$\omega^2 = \frac{4(m+M)gl\sin\theta}{2ml^2\sin\theta\cos\theta} = \frac{2(m+M)g}{ml\cos\theta}$$

smallest stable frequency is for $\cos\theta = 1$, $\theta = 0$

$$\omega_c^2 = \frac{2(m+M)g}{ml}$$

d) $\omega^2 > \omega_c^2 \rightarrow \omega^2 = \dfrac{2(m+M)g}{m\ell\cos\theta}$

θ^* = new stable equilibrium location

$\cos\theta^* = \dfrac{\omega_c^2}{\omega^2} \rightarrow \theta^* = \cos^{-1}\left(\dfrac{\omega_c^2}{\omega^2}\right)$

for small oscillations, $\theta \rightarrow \theta^* + \varepsilon$

$\sin(\theta^*+\varepsilon) = \sin\theta^*\cos\varepsilon + \sin\varepsilon\cos\theta^* = \sin\theta^* + \varepsilon\cos\theta^*$

$\cos(\theta^*+\varepsilon) = \cos\theta^* - \varepsilon\sin\theta^*$

$\sin(\theta^*+\varepsilon)\cos(\theta^*+\varepsilon) = \sin\theta^*\cos\theta^* - \varepsilon\sin^2\theta^* + \varepsilon\cos^2\theta^* - \cos\theta^*\sin\theta^*\varepsilon^2$

$\sin^2(\theta^*+\varepsilon) = \sin^2\theta^* + \varepsilon^2\cos^2\theta^* + 2\sin\theta^*\cos\theta^*\varepsilon$

only want terms of order ε.

Also $\ddot{\theta} = \ddot{\varepsilon}$, $\dot{\theta} = \dot{\varepsilon}$

The E-L eqns become:

$$2m\ell^2\omega^2\left(\sin\theta^*\cos\theta^* - \varepsilon(\sin^2\theta^* - \cos^2\theta^*)\right) - 4M\ell^2\dot{\varepsilon}^2\left(\sin\theta^*\cos^* - \varepsilon(\sin^2\theta^* - \cos^2\theta^*)\right)$$
$$+ 4(m+M)g\ell(\sin\theta^* + \cos\theta^*\varepsilon) = \ddot{\varepsilon}\left(2m\ell^2 + 4M\ell^2(\sin^2\theta^* + 2\varepsilon\sin\theta^*\cos\theta^*)\right)$$

drop terms of $\dot{\varepsilon}\varepsilon$, $\ddot{\varepsilon}\varepsilon$, $\dot{\varepsilon}^2$, $\ddot{\varepsilon}^2$; we have

$$-2m\ell^2\omega^2(\sin^2\theta^* - \cos^2\theta^*)\varepsilon + 4(m+M)g\ell\cos\theta^*\varepsilon + 2m\ell^2\omega^2\sin\theta^*\cos\theta^*$$
$$+ 4(m+M)g\ell\sin\theta^* = \ddot{\varepsilon}(2m\ell^2 + 4M\ell^2\sin^2\theta^*)$$

note: $m\ell\omega^2\cos\theta = 2(m+M)g$ so our two const. terms add. We now have

$$\varepsilon + \underbrace{\left[\frac{2m\ell^2 + 4M\ell^2\sin^2\theta^*}{2m\ell^2\omega^2(\sin^2\theta^* - \cos^2\theta^*) - 4(m+M)g\ell\cos\theta^*}\right]}_{\omega'^2 \text{ of small oscillations around } \theta^*.}\ddot{\varepsilon} = \underbrace{\frac{8(m+M)g\ell\sin\theta^*}{2m\ell^2\omega^2(\sin^2\theta^* - \cos^2\theta^*) - 4(m+M)g\ell\cos\theta^*}}_{\text{Const. of integration.}}$$

e) P_ϕ is conserved because we have a constant rotation.

Total Energy is conserved.

2. $L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2) - V(x+ky+lz, y+a\phi)$

a) $\bar{x} + k\bar{y} + l\bar{z} = x + ky + lz$

$\bar{y} + a\bar{\phi} = y + a\phi$

$\bar{\phi} = \phi \pm \lambda$

$\bar{y} = y \mp a\lambda$

$\begin{cases} \bar{x} = x + ka\lambda \\ \bar{z} = z \end{cases}$ or $\begin{cases} \bar{x} = x \\ \bar{z} = z + \frac{ka\lambda}{l} \end{cases}$

we have 4 families of transformations

b) depending on the choice of the above transformation, we will have 3 continuous symmetries at a time: ϕ, y, x or ϕ, y, z

$Q_\phi = m\rho^2\dot{\phi}(\pm 1)$

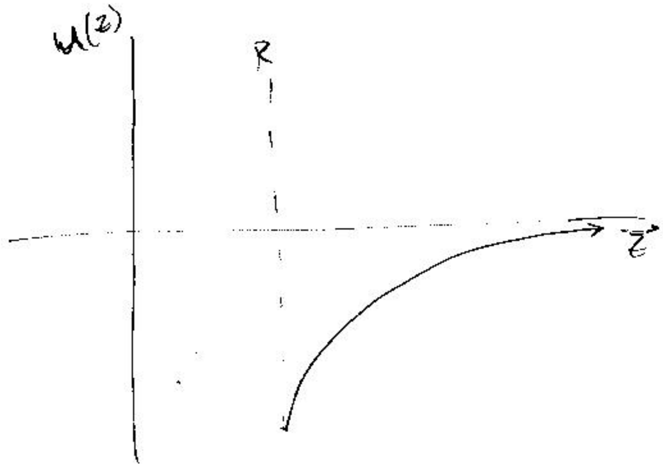
$Q_y = m\dot{y}(\mp a)$

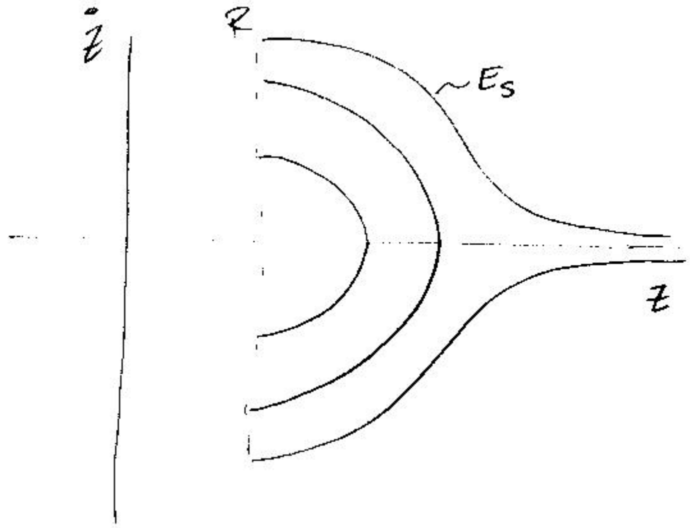
$Q_x = (m\dot{x})(ka)$ or $Q_z = m\dot{z}\left(\frac{ka}{l}\right)$

c) The hamiltonian is also conserved: $H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$

3. $L = \frac{1}{2} m \dot{z}^2 + \frac{GMm}{R+z}$

a) $U = -\frac{GMm}{R+z}$





b) The separatrix has energy equal to the escape velocity energy which is at $E_s = \frac{GMm}{R}$

c) $H = \dot{q}_1 \left(\frac{\partial L}{\partial \dot{q}_1} \right) - L = \frac{1}{2} m \dot{z}^2 - \frac{GMm}{R+z}$

4. $$L = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} - U(x)$$

a) $$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x}$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}\left(1-\frac{\dot{x}^2}{c^2}\right)^{-1/2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{m\ddot{x}}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} - \frac{m\dot{x}^2}{c^2}\frac{\ddot{x}}{\left(1-\frac{\dot{x}^2}{c^2}\right)^{3/2}}$$

$$= m\ddot{x}\left(\frac{c^2}{\sqrt{c^2-\dot{x}^2}} - \frac{c^2\dot{x}^2}{(c^2-\dot{x}^2)^{3/2}}\right)$$

$$\Rightarrow m\ddot{x}\left(\frac{c^2(c^2-2\dot{x}^2)}{(c^2-\dot{x}^2)^{3/2}}\right) + \frac{\partial U}{\partial x} = 0$$

b) Total energy is conserved $\left(\frac{\partial H}{\partial t} = 0\right)$

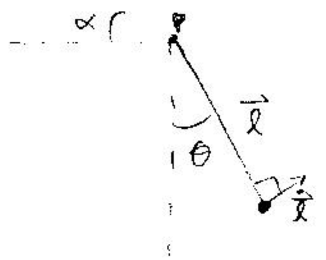
c) $$\ddot{x} + \omega^2\frac{\partial U}{\partial x} = 0$$

$$\omega^2 = \frac{(c^2-\dot{x}^2)^{3/2}}{mc^2(c^2-2\dot{x}^2)}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{mc^2(c^2-2\dot{x}^2)}{(c^2-\dot{x}^2)^{3/2}}}$$

5. $\vec{v} = at$

a)



$$\dot{\vec{l}} = \vec{v} - \vec{a}t$$

$$\vec{v} = \dot{\vec{l}} + \vec{a}t$$

$$\vec{v}^2 = \vec{v}\cdot\vec{v} = \dot{l}^2 + (at)^2 + 2\vec{a}t\cdot\dot{\vec{l}}$$

$$= \dot{l}^2 + a^2t^2 + 2at\dot{l}\cos\phi$$

$$\phi = \alpha + \theta$$

$$\vec{v}^2 = \dot{l}^2 + a^2t^2 + 2at\dot{l}\cos(\alpha+\theta)$$

where $\dot{\vec{l}} = \cancel{\dot{l}\hat{r}}^{\to 0} + l\dot{\theta}\hat{\theta}$

$$v^2 = l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\cos(\alpha+\theta)$$

$T = \frac{1}{2}mv^2 - U$, $U = -mgl\cos\theta + \frac{1}{2}mgat^2\sin\alpha$

$$L = \frac{1}{2}m\left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\cos(\alpha+\theta)\right) + mgl\cos\theta - \frac{1}{2}mgat^2\sin\alpha$$

b) $$\frac{\partial L}{\partial \theta} = -matl\dot{\theta}\sin(\alpha+\theta) - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + matl\cos(\alpha+\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + mal\cos(\alpha+\theta) - matl\sin(\alpha+\theta)\dot{\theta}$$

eqn of motion: $-mgl\sin\theta = ml^2\ddot{\theta} + mal\cos(\alpha+\theta)$

c)

$$m\ell^2\ddot{\theta} + ma\ell(\cos\alpha\cos\theta - \sin\alpha\sin\theta) + mg\ell\sin\theta = 0$$

$$m\ell^2\ddot{\theta} + ma\ell(\cos\alpha - \theta\sin\alpha) + mg\ell\theta = 0$$

$$m\ell^2\ddot{\theta} + \theta(mg\ell - ma\ell\sin\alpha) = -ma\ell\cos\alpha$$

$$\omega^2 = \frac{mg\ell - ma\ell\sin\alpha}{m\ell^2} = \frac{g - a\sin\alpha}{\ell}$$