Solution Set 3

1) In N-D space, the distance element ds is

$$ds = \sqrt{\sum_i^n dx_i^2} \quad \text{so we have}$$

$$S = \int_{t_1}^{t_2} dt \sqrt{\sum_{i=1}^n \dot{x}_i^2} \quad . \quad \text{Define } L = \sqrt{\sum_{i=1}^n \dot{x}_i^2} \quad .$$

Applying the Euler-Lagrange eqn, we see that $\frac{\partial L}{\partial x_i} = 0$. Therefore, in order to extremize the distance, we must satisfy the condition:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right) = 0 \quad \therefore \quad \frac{\partial L}{\partial \dot{x}_j} = \frac{\dot{x}_j}{\sqrt{\sum_{i=1}^n \dot{x}_i^2}} = C_j$$

$\therefore$ we have a constant velocity which can only be satisfied for a straight line. To see this, we have

$$\frac{\dot{x}_j}{C_j} = \sqrt{\sum_{i=1}^n \dot{x}_i^2} \quad , \text{integrate from } t \to t_1$$

and we have: $\frac{x_j - x_{j1}}{C_j} = \frac{x_k - x_{k_1}}{C_k} = \frac{x_\ell - x_{\ell_1}}{C_\ell} = \cdots = \frac{x_n - x_{n_1}}{C_n}$

integrate from $t_1 \to t_2$:

$$\frac{x_{j2} - x_{j1}}{C_j} = \frac{x_{k_2} - x_{k_1}}{C_k} = \cdots = \frac{x_{n_2} - x_{n_1}}{C_n}$$

solving for all C_i, we get

$$\frac{x_j - x_{j_1}}{x_{j_2} - x_{j_1}} = \frac{x_k - x_{k_1}}{x_{k_2} - x_{k_1}} = \cdots = \frac{x_n - x_{n_1}}{x_{n_2} - x_{n_1}}$$

which describes a straight line path in n-D space.

2. a) $\rho(\phi)$, $z(\rho(\phi))$

$$D[\rho(\phi)] = \int \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2} = \int d\phi \sqrt{\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2 + \left(\frac{dz}{d\rho}\frac{d\rho}{d\phi}\right)^2}$$

$$= \int d\phi \sqrt{\rho^2 + [1 + z'(\rho)^2]\, \rho'^2(\phi)}$$

$$\frac{\partial L}{\partial \rho} - \frac{d}{d\phi}\left(\frac{\partial L}{\partial \rho'}\right) = 0 \qquad , L(\rho, \rho', \phi) = \sqrt{\rho^2 + [1 + z'^2(\rho)]\, \rho'^2(\phi)}$$

$$\frac{\partial L}{\partial \rho} = \frac{\rho + \rho'^2 z'(\rho) z''(\rho)}{\sqrt{\rho^2 + [1 + z'^2(\rho)]\rho'^2(\phi)}}$$

$$\frac{\partial L}{\partial \rho'} = \frac{\rho' [1 + z'^2(\rho)]}{\sqrt{\rho^2 + [1 + z'^2(\rho)]\rho'^2(\phi)}}$$

$$\frac{d}{d\phi}\left(\frac{\partial L}{\partial \rho'}\right) = \frac{2 z'(\rho) z''(\rho) \rho'^2 + (1 + z'^2(\rho))\rho''}{\sqrt{\rho^2 + [1 + z'^2(\rho)]\, \rho'^2(\phi)}} + \frac{(1 + z'^2(\rho))\rho'^2 [\rho + z'(\rho) z''(\rho)\rho'^2 + (1 + z'^2(\rho))\rho'']}{[\rho^2 + [1 + z'^2(\rho)]\rho'^2(\phi)]^{3/2}}$$

phew

b) $\mathbf{J} = \rho' \frac{\partial L}{\partial \rho'} + L = \text{constant}$

$$J = \frac{\rho'^2 [1 + z'^2(\rho)]}{\sqrt{\rho^2 + [1 + z'^2(\rho)]\rho'^2(\phi)}} - \sqrt{\rho^2 + [1 + z'^2(\rho)]\rho'^2(\phi)} = -\frac{\rho^2}{\sqrt{\rho^2 + [1 + z'^2(\rho)]\rho'^2(\phi)}}$$

c) Eq 6.39: $d\phi = \frac{a}{\rho}\sqrt{\frac{1 + z'^2}{\rho^2 - a^2}}\, d\rho$

solve J for ρ': $\rho' = \frac{\rho\,(\rho^2 - J^2)^{1/2}}{J\,(1 + z'^2)^{1/2}} \rightarrow \frac{d\phi}{d\rho} = \frac{J}{\rho}\sqrt{\frac{1 + z'^2}{\rho^2 - J^2}}$

and we see the eqns are equivalent.

3. $ds^2 = dx^2 + dy^2 \rightarrow dt^2 = \frac{dx^2 + dy^2}{dv^2}$, $dv = \frac{c}{n(y)}$

$T[y(x)] = \frac{1}{c}\int dx\, n(y)\sqrt{1+y'^2}$

$L(y, y', x) = n(y)\sqrt{1+y'^2}$, $\frac{\partial L}{\partial x} = 0$, $\therefore$ J is const.

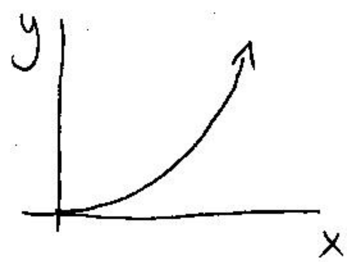
$$J = y' \frac{dL}{dy'} - L = -\frac{n(y)\, y'^2}{\sqrt{1+y'^2}} = n_0$$

$$\Rightarrow \boxed{dy = \sqrt{\left(\frac{n(y)}{n_0}\right)^2 - 1}\; dx}$$

a) $n = ay \rightarrow dy = \sqrt{\left(\frac{ay}{n_0}\right)^2 - 1}\; dx$ let $\frac{ay}{n_0} = z$, $dy = \frac{n_0}{a} dz$

$$\frac{n_0}{a}\frac{dz}{\sqrt{z^2-1}} = dx \rightarrow x = \frac{n_0}{a}\cosh^{-1} z$$

$$\therefore y = \frac{n_0}{a}\cosh\frac{ax}{n_0}$$

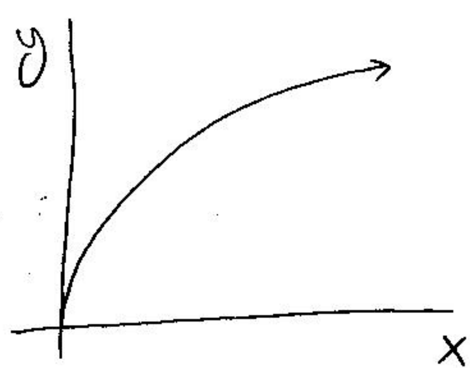


b) $dy = \sqrt{\left(\frac{a}{n_0 y}\right)^2 - 1}\; dx$, let $\frac{n_0 y}{a} = z$, $dy = \frac{a}{n_0} dz$

$$\frac{a}{n_0}\frac{dz\; z^2}{\sqrt{1-z^2}} = dx \rightarrow \frac{a}{n_0} z^2 d\sin^{-1}(z) = dx$$

$\frac{a}{n_0} z^2 \sin^{-1}(z) = x$

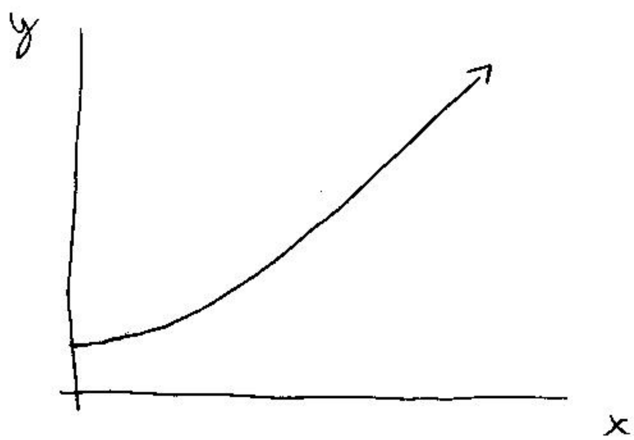
$\frac{n_0}{a} y^2 \sin^{-1}\left(\frac{n_0}{a} y\right) = x$



3. c) $dy = \sqrt{\left(\frac{a}{n_o}\right)^2 y - 1}\, dx \qquad z^2 = y\left(\frac{a}{n_o}\right)^2, \quad dy = \left(\frac{n_o}{a}\right)^2 2z\,dz$

$$2\left(\frac{n_o}{a}\right)^2 \frac{z\,dz}{\sqrt{z^2-1}} = dx \quad \rightarrow \quad 2\left(\frac{n_o}{a}\right)^2 z\, d\cosh^{-1}(z) = x$$

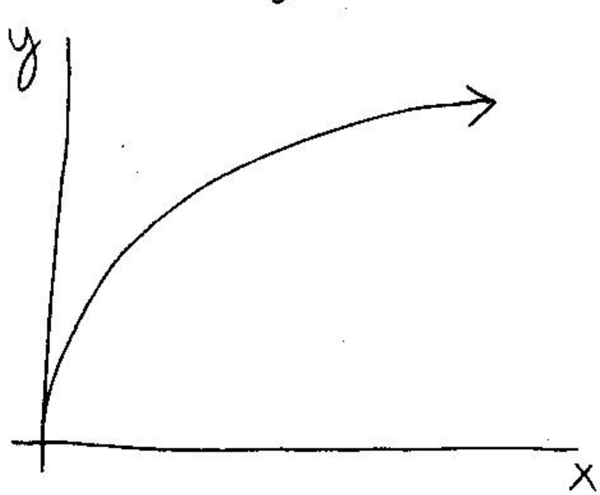
$$2\,\frac{n_o}{a}\sqrt{y}\,\cosh^{-1}\left(\sqrt{y}\,\frac{a}{n_o}\right) = x$$



d) $dy = \sqrt{\left(\frac{a}{n_o}\right)^2 \frac{1}{y} - 1}\, dx, \qquad \frac{1}{z^2} = y\left(\frac{n_o}{a}\right)^2, \quad dy = -\left(\frac{a}{n_o}\right)^2 \frac{2}{z^3} dz$

$$-2\left(\frac{a}{n_o}\right)^2 \frac{dz}{z^3\sqrt{z^2-1}} = dx \quad \rightarrow 2\left(\frac{a}{n_o}\right)^2 \frac{1}{z^2}\, d\csc^{-1}(z) = dx$$

$$2y\csc^{-1}\left(\frac{1}{\sqrt{y}}\frac{a}{n_o}\right) = x$$



4.

a) $U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2$

$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2$

$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + m_2 \dot{x}_1 \dot{x}_2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 x_2^2$

b) $\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) = \frac{\partial L}{\partial q_i}$

x_1: $(m_1 + m_2)\ddot{x}_1 + m_2 \ddot{x}_2 = -k_1 x_1$

x_2: $m_2 \ddot{x}_1 + m_2 \ddot{x}_2 = -k_2 x_2$

c) $\left(1 + \frac{m_2}{m_1}\right)\ddot{x}_1 + \frac{m_2}{m_1}\ddot{x}_2 + \omega_1^2 x_1 = 0$

$\ddot{x}_1 + \ddot{x}_2 + \omega_2^2 x_2 = 0$

where $\omega_1^2 = \frac{k_1}{m_1}$, $\omega_2^2 = \frac{k_2}{m_2}$

$x_1(t) = x_1 e^{-i\omega t}$, $x_2(t) = x_2 e^{-i\omega t}$ ve have

$$\underbrace{\begin{pmatrix} -\omega^2 - \omega^2 \frac{m_2}{m_1} + \omega_1^2 & -\omega^2 \frac{m_2}{m_1} \\ -\omega^2 & -\omega^2 + \omega_2^2 \end{pmatrix}}_{\det = 0} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\therefore \omega^2 = \frac{m_1}{2m_2}\left[\omega_2^2\left(2 + \frac{m_2}{m_1}\right) + \omega_1^2 \pm \sqrt{\left(\omega_2^2\left(2 + \frac{m_2}{m_1}\right) + \omega_1^2\right)^2 - 4\omega_1^2\omega_2^2\frac{m_2}{m_1}}\right]$$

5. $U(x,y) = U_0 \ln\left(\frac{x^2+y^2}{a^2}\right)$

a) $x = r\cos\theta$, $y = r\sin\theta$

$$U(r) = U_0 \ln\left(\frac{r}{a}\right)^2 \rightarrow U(r) = 2U_0 \ln\frac{r}{a}$$

for $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, $U = 2U_0 \ln\frac{r}{a}$

$$L(r,\dot{r},\theta,\dot{\theta},t) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - 2U_0 \ln\frac{r}{a}$$

b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$

r: $m\ddot{r} = mr\dot{\theta}^2 - \frac{2U_0}{r}$

θ: $mr^2\ddot{\theta} = 0$

c) • We see that $P_\theta = \frac{\partial L}{\partial \dot{\theta}}$ is a conserved quantity.

• Also, because L is not explicity dependent on time, we expect the hamiltonian to be conserved. ie, we have a conservation of energy.

$$H = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2 - r^2\dot{\theta}^2) + U$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + 2U_0 \ln\frac{r}{a}$$