Solution Set # 2

1) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

in general we have $x(t) = Ae^{rt}$

$\Rightarrow r^2 + 2\beta r + \omega_0^2 = 0 \quad \Rightarrow \quad r = \dfrac{-2\beta \pm \sqrt{4\beta^2 - \omega_0^2}}{2}$

for $\omega_0 = \beta$, $r = -\beta$ where r now becomes a double root. The general sol'n to a double root is $x(t) = Ae^{rt} + Bte^{rt}$

where we have $x(t) = Ce^{-\beta t} + Dte^{-\beta t}$

$$x(t) = (C + Dt)e^{-\beta t}$$

2) $\dddot{x} + (2\beta + \gamma)\ddot{x} + (\omega_0^2 + 2\beta\gamma)\dot{x} + \gamma\omega_0^2 x = f_0 \cos(\Omega t)$

$$L = \left(\frac{d}{dt} + \gamma\right)\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)$$

$\therefore$ we have $X(t) = Ae^{-\gamma t} + Be^{-\beta + \sqrt{\beta^2 - \omega_0^2}} + Ce^{\beta - \sqrt{\beta^2 - \omega_0^2}}$

$X_1(t) = f_0 \cos(\Omega t + \delta\Omega) = \mathrm{Re}\left[A(\Omega)e^{i\delta(\Omega)} f_0 e^{-i\Omega t}\right]$

where we should have two phase terms for our solution. $\therefore$

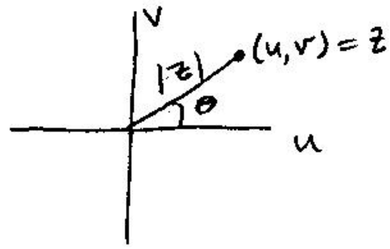
$$X_1(t) = X_0 e^{-i\Omega t} e^{i\delta_1(\Omega)} e^{i\delta_2(\Omega)}$$

$\Rightarrow \left(\frac{d}{dt} + \gamma\right)\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right) X_0 e^{-i\Omega t} = f_0 e^{-i\Omega t} e^{i\delta_1(\Omega)} e^{i\delta_2(\Omega)}$

$$\underbrace{(\gamma - i\Omega)}_{z_1}\underbrace{(-\Omega^2 - i2\beta\Omega + \omega_0^2)}_{z_2}\, x_0 e^{-i\Omega t} e^{i\delta_1(\Omega)} e^{i\delta_2(\Omega)} = f_0 e^{-i\Omega t} e^{i\delta(\Omega)} e^{i\delta_2(\Omega)}$$

recall that in general, a complex # z can be represented as $z = |z| e^{i\theta}$ where $z = u + iv$

$|z| = \sqrt{u^2 - v^2}$ and



$\theta = \tan^{-1}\left(\frac{v}{u}\right)$

in our case, $z_1 = \gamma - i\Omega = \sqrt{\gamma^2 + \Omega^2}\, e^{i\delta_1(\Omega)}$

where $\delta_1(\Omega) = \tan^{-1}\left(\frac{\Omega}{\gamma}\right)$

and $z_2 = (\omega_0^2 - \Omega^2 - i2\beta\Omega) = \sqrt{(\omega_0^2 - \beta^2)^2 + 4\beta^2\Omega^2}\, e^{i\delta_2(\Omega)}$

where $\delta_2(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right)$

To find the amplitude, we want the real part, s.t.

$$X_0 = \frac{f_0}{|z_1||z_2|} = \frac{f_0}{\left[(\gamma^2 + \Omega^2)((\omega_0^2 - \Omega^2)^2 + 4\beta^2\omega_0^2)\right]^{1/2}}$$

and $A(\Omega) = \left[(\gamma^2 + \Omega^2)((\omega_0 - \Omega)^2 + 4\beta^2\omega_0^2)\right]^{-1/2}$

$$\delta(\Omega) = \tan^{-1}\left(\frac{\Omega}{\gamma}\right) + \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right)$$

To find $A(\Omega)_{max}$, let $\frac{\partial A(\Omega)}{\partial \Omega} = 0$. We find

$$\Omega = \pm \frac{1}{\sqrt{3}}\left[-4\beta^2 - \gamma^2 + 2\omega_0^2 \pm \left(16\beta^4 + (\gamma^2 + \omega_0^2)^2 - 4\beta^2(\gamma^2 + 4\omega^2)\right)^{1/2}\right]^{1/2}$$

3) $f(t) = f_0 e^{-\gamma t} \sin(\Omega t)\, \Theta(t)$, $\Theta(t) \equiv \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}$

$X(t) = \int_{-\infty}^{t} F(t')\, G(t-t')\, dt'$ where for a damped oscillator we have

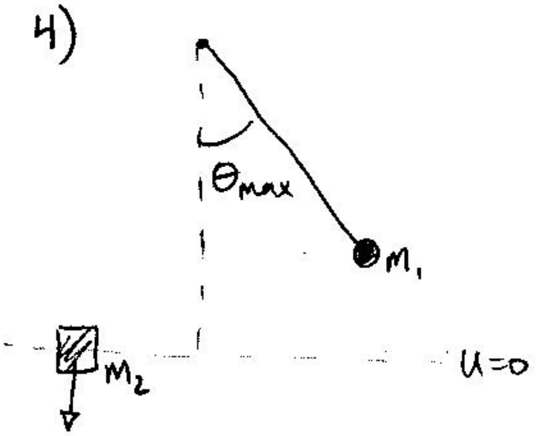
$$G(t-t') \equiv \begin{cases} \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin(\omega_1 (t-t')) & t > t_0 \\ 0 & t < t_0 \end{cases}$$

Therefore,

$$x(t) = \frac{f_0}{m\omega_1} \int_{t_0}^{t} e^{-\beta(t-t')} \sin(\omega_1(t-t'))\, e^{-\gamma t'} \sin(\Omega t')\, dt'$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$

4)



$$Q \equiv 2\pi \frac{E_{Tot}}{\Delta E}\bigg|_{1\ period} = \frac{\omega_r}{\beta}$$

$E_{initial} = E_{Tot} = m_1 g \ell (1 - \cos\theta_{max})$

$E_{final} = m_1 g \ell (1 - \cos\theta_{max}) - m_2 g \Delta h$

$\Delta E = m_2 g \Delta h$, where $\Delta h = \bar{v} \cdot T$

$T = \frac{2\pi}{\omega}$, $\omega \approx \sqrt{\frac{g}{\ell}}$, $\bar{v} = \frac{.8}{7 \times 24 \times 3600} = m/s$

$$Q = \cancel{2\pi} \frac{m_1 \cancel{g} \ell (1 - \cos\theta_{max})}{m_2 \cancel{g} (.8 \cdot \cancel{2\pi} \sqrt{\frac{\ell}{g}})} \cdot (7 \times 24 \times 3600)$$

$$Q = \frac{m_1}{m_2} \frac{(1 - \cos\theta_{max}) \sqrt{\ell \cdot g}}{.8} \cdot (7 \times 24 \times 3600) \approx 178.194$$

5. $\dot{x} + \gamma x = F(t)$, $x(t) = x_h(t) + \int_{-\infty}^{t} G(t-t') F(t') dt'$

$L = \left(\frac{d}{dt} + \gamma\right) \rightarrow Lx = 0 \quad \therefore \quad x_h(t) = Ae^{-\gamma t}$

Now, $L \cdot x(t) = F(t)$

$\Rightarrow x(t) = L^{-1} F(t)$ where we can define the inverse operator to be the Green's fcn s.t.

$$x(t) = L^{-1} F(t) = \int^{t} G(t-t') F(t') dt'$$

Using the definition of the delta fcn,

$\int_{-\infty}^{\infty} \delta(t-t') F(t') dt' = F(t)$. We see

$L \cdot G(t-t') = \delta(t-t')$ where a short proof shows:

$$L x(t) = L \int_{-\infty}^{\infty} G(t-t') F(t) dt$$

$$= \int_{-\infty}^{\infty} L G(t-t') F(t') dt'$$

$$= \int_{-\infty}^{\infty} \delta(t-t') F(t') dt' = F(t) \checkmark$$

Here, we have

$$L \cdot G(t-t') = \delta(t-t')$$

$$\left(\frac{d}{dt} + \gamma\right) G(t-t') = \delta(t-t') = \begin{cases} 0 & t \neq t' \\ 1 & t = t' \end{cases}$$

where $G(t-t') = e^{-\gamma(t-t')}$

$$\therefore \quad x(t) = Ae^{-\gamma t} + \int_{t_0}^{t} e^{-\gamma(t-t')} F(t') dt'$$