

**PHYSICS 110A : CLASSICAL MECHANICS
MIDTERM EXAM #1**

[1] A particle of mass m moves in the one-dimensional potential

$$U(x) = U_0 \frac{x^2}{a^2} e^{-x/a} . \quad (1)$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \rightarrow \pm\infty$.

(b) Sketch a representative set of phase curves. Identify and classify any and all fixed points. Find the energy of each and every separatrix.

(c) Sketch all the phase curves for motions with total energy $E = \frac{2}{5} U_0$. Do the same for $E = U_0$. (Recall that $e = 2.71828\dots$)

(d) Derive an expression for the period T of the motion when $|x| \ll a$.

Solution:

(a) Clearly $U(x)$ diverges to $+\infty$ for $x \rightarrow -\infty$, and $U(x) \rightarrow 0$ for $x \rightarrow +\infty$. Setting $U'(x) = 0$, we obtain the equation

$$U'(x) = \frac{U_0}{a^2} \left(2x - \frac{x^2}{a} \right) e^{-x/a} = 0 , \quad (2)$$

with (finite x) solutions at $x = 0$ and $x = 2a$. Clearly $x = 0$ is a local minimum and $x = 2a$ a local maximum. Note $U(0) = 0$ and $U(2a) = 4 e^{-2} U_0 \approx 0.541 U_0$.

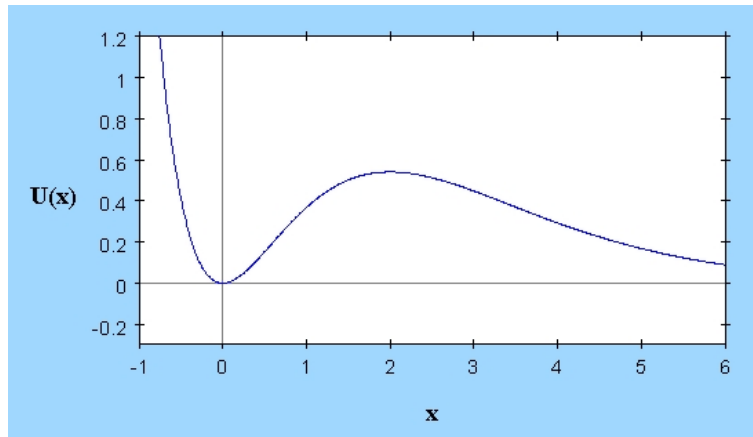


Figure 1: The potential $U(x)$. Distances are here measured in units of a , and the potential in units of U_0 .

(b) Local minima of a potential $U(x)$ give rise to centers in the (x, v) plane, while local maxima give rise to saddles. In Fig. 2 we sketch the phase curves. There is a center at $(0, 0)$ and a saddle at $(2a, 0)$. There is one separatrix, at energy $E = U(2a) = 4 e^{-2} U_0 \approx 0.541 U_0$.

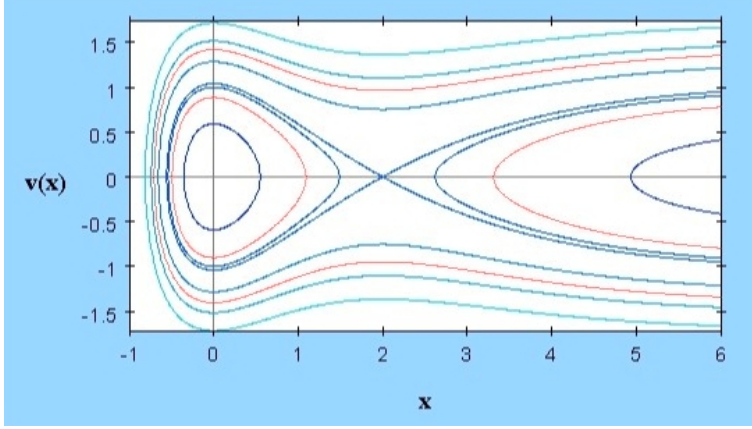


Figure 2: Phase curves for the potential $U(x)$. The red curves show phase curves for $E = \frac{2}{5} U_0$ (interior, disconnected red curves, $|v| < 1$) and $E = U_0$ (outlying red curve). The separatrix is the dark blue curve which forms a saddle at $(x, v) = (2, 0)$, and corresponds to an energy $E = 4e^{-2} U_0$.

(c) Even without a calculator, it is easy to verify that $4e^{-2} > \frac{2}{5}$. One simple way is to multiply both sides by $\frac{5}{2}e^2$ to obtain $10 > e^2$, which is true since $e^2 < (2.71828\dots)^2 < 10$. Thus, the energy $E = \frac{2}{5} U_0$ lies below the local maximum value of $U(2a)$, which means that there are *two* phase curves with $E = \frac{2}{5} U_0$.

It is also quite obvious that the second energy value given, $E = U_0$, lies above $U(2a)$, which means that there is a single phase curve for this energy. One finds bound motions only for $x < 2$ and $0 \leq E < U(2a)$. The phase curves corresponding to total energy $E = \frac{2}{5} U_0$ and $E = U_0$ are shown in Fig. 2.

(d) Expanding $U(x)$ in a Taylor series about $x = 0$, we have

$$U(x) = \frac{U_0}{a^2} \left\{ x^2 - \frac{x^3}{a} + \frac{x^4}{2a^2} + \dots \right\}. \quad (3)$$

The leading order term is sufficient for $|x| \ll a$. The potential energy is then equivalent to that of a spring, with spring constant $k = 2U_0/a^2$. The period is

$$\boxed{T = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{ma^2}{2U_0}}}. \quad (4)$$

[2] A forced, damped oscillator obeys the equation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega_0 t) . \quad (5)$$

You may assume the oscillator is underdamped.

(a) Write down the most general solution of this differential equation.

(b) Your solution should involve two constants. Derive two equations relating these constants to the initial position $x(0)$ and the initial velocity $\dot{x}(0)$. *You do not have to solve these equations.*

(c) Suppose $\omega_0 = 5.0 \text{ s}^{-1}$, $\beta = 4.0 \text{ s}^{-1}$, and $f_0 = 8 \text{ cm s}^{-2}$. Suppose further you are told that $x(0) = 0$ and $x(T) = 0$, where $T = \frac{\pi}{6} \text{ s}$. Derive an expression for the initial velocity $\dot{x}(0)$.

Solution: (a) The general solution with forcing $f(t) = f_0 \cos(\Omega t)$ is

$$x(t) = x_h(t) + A(\Omega) f_0 \cos(\Omega t - \delta(\Omega)) , \quad (6)$$

with

$$A(\Omega) = \left[(\omega_0^2 - \Omega^2)^2 + 4\beta^2 \Omega^2 \right]^{-1/2} , \quad \delta(\Omega) = \tan^{-1} \left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2} \right) \quad (7)$$

and

$$x_h(t) = C e^{-\beta t} \cos(\nu t) + D e^{-\beta t} \sin(\nu t) , \quad (8)$$

with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

In our case, $\Omega = \omega_0$, in which case $A = (2\beta\omega_0)^{-1}$ and $\delta = \frac{1}{2}\pi$. Thus, the most general solution is

$$x(t) = C e^{-\beta t} \cos(\nu t) + D e^{-\beta t} \sin(\nu t) + \frac{f_0}{2\beta\omega_0} \sin(\omega_0 t) . \quad (9)$$

(b) We determine the constants C and D by the boundary conditions on $x(0)$ and $\dot{x}(0)$:

$$\boxed{x(0) = C} , \quad \boxed{\dot{x}(0) = -\beta C + \nu D + \frac{f_0}{2\beta}} . \quad (10)$$

Thus,

$$C = x(0) , \quad D = \frac{\beta}{\nu} x(0) + \frac{1}{\nu} \dot{x}(0) - \frac{f_0}{2\beta\nu} . \quad (11)$$

(c) From $x(0) = 0$ we obtain $C = 0$. The constant D is then determined by the condition at time $t = T = \frac{1}{6}\pi$.

Note that $\nu = \sqrt{\omega_0^2 - \beta^2} = 3.0 \text{ s}^{-1}$. Thus, with $T = \frac{1}{6}\pi$, we have $\nu T = \frac{1}{2}\pi$, and

$$x(T) = D e^{-\beta T} + \frac{f_0}{2\beta\omega_0} \sin(\omega_0 T) . \quad (12)$$

This determines D :

$$D = -\frac{f_0}{2\beta\omega_0} e^{\beta T} \sin(\omega_0 T) . \quad (13)$$

We now can write

$$\dot{x}(0) = \nu D + \frac{f_0}{2\beta} \quad (14)$$

$$= \frac{f_0}{2\beta} \left(1 - \frac{\nu}{\omega_0} e^{\beta T} \sin(\omega_0 T) \right) \quad (15)$$

$$= \boxed{\left(1 - \frac{3}{10} e^{2\pi/3} \right) \text{ cm/s}} . \quad (16)$$

Numerically, the value is $\dot{x}(0) \approx 0.145 \text{ cm/s}$.