PHYSICS 110A : CLASSICAL MECHANICS
FINAL EXAM SOLUTIONS

[1] Two blocks and three springs are configured as in Fig. 1. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

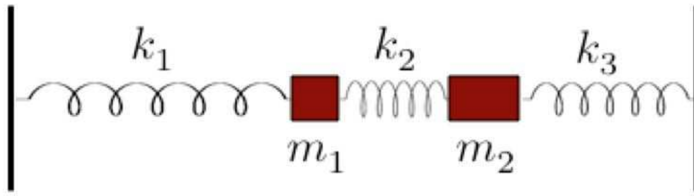

Figure 1: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.
`[5 points]`

(b) Find the T and V matrices.
`[5 points]`

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.
`[5 points]`

(d) Find the normal modes of oscillation.
`[5 points]`

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.
`[5 points]`

Solution

(a) The Lagrangian is

$$\boxed{L = \tfrac{1}{2}m_1\, \dot{x}_1^2 + \tfrac{1}{2}m_2\, \dot{x}_2^2 - \tfrac{1}{2}k_1\, x_1^2 - \tfrac{1}{2}k_2\, (x_2 - x_1)^2 - \tfrac{1}{2}k_3\, x_2^2}$$

(b) The T and V matrices are

$$\boxed{\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i \, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} \quad , \quad \boxed{\mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i \, \partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda \, \omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} \ .$$

The determinant is

$$\begin{aligned} \det \left(\omega^2 \mathrm{T} - \mathrm{V} \right) &= \left(2\lambda^2 - 11\lambda + 14 \right) k^2 \\ &= (2\lambda - 7) \, (\lambda - 2) \, k^2 \ . \end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\boxed{\omega_- = \sqrt{\frac{2k}{m}}} \quad , \quad \boxed{\omega_+ = \sqrt{\frac{7k}{2m}}}$$

(d) The normal modes are determined from $(\omega_a^2 \mathrm{T} - \mathrm{V}) \, \vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

The standard normalization $\psi_i^{(a)} \, \mathrm{T}_{ij} \, \psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \quad , \quad \mathcal{C}_2 = \frac{1}{\sqrt{6m}} \ . \tag{1}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) \ .$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3}b \quad , \quad B = \tfrac{1}{3}b \quad , \quad C = 0 \quad , \quad D = 0 \; .$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3}b \cdot \Big(2\cos(\omega\_t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3}b \cdot \Big(\cos(\omega\_t) - \cos(\omega_+ t) \Big) \; . \end{aligned}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega\_t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega\_t = \omega_+ t - \pi \quad \Rightarrow \quad \boxed{t^* = \frac{2\pi}{\omega_- + \omega_+}}$$

[2] Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) ,$$

where b is a constant with dimensions of length.

(a) For what values of the relative angular momentum ℓ does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?
`[7 points]`

(c) For the case where a circular orbit exists, sketch the phase curves for the radial motion in the $(r, \dot{r})$ half-plane. Identify the energy ranges for bound and unbound orbits.
`[5 points]`

(c) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $|\eta| \ll r_0$. Find the equation for the shape $\eta(\phi)$ of the perturbation.
`[8 points]`

(d) What is the angle $\Delta\phi$ through which periapsis changes each cycle? For which value(s) of ℓ does the perturbed orbit not precess?
`[5 points]`

Solution

(a) The effective potential is

$$\begin{aligned} U_{\text{eff}}(r) &= \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^2} + U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) . \end{aligned}$$

where $\mu = m_1 m_2/(m_1 + m_1)$ is the reduced mass. For a circular orbit, we must have $U'_{\text{eff}}(r) = 0$, or

$$\frac{l^2}{\mu r^3} = U'(r) = \frac{2rU_0 b^2}{r^2\,(r^2 + b^2)} .$$

The solution is

$$\boxed{r_0^2 = \frac{b^2 \ell^2}{2\mu b^2 U_0 - \ell^2}}$$

Since $r_0^2 > 0$, the condition on ℓ is

$$\boxed{\ell < \ell_{\text{c}} \equiv \sqrt{2\mu b^2 U_0}}$$

For large r, we have

$$U_{\text{eff}}(r) = \left(\frac{\ell^2}{2\mu} - U_0\, b^2\right) \cdot \frac{1}{r^2} + \mathcal{O}(r^{-4}) \ .$$

Thus, for $\ell < \ell_c$ the effective potential is negative for sufficiently large values of r. Thus, over the range $\ell < \ell_c$, we must have $U_{\text{eff,min}} < 0$, which must be a global minimum, since $U_{\text{eff}}(0^+) = \infty$ and $U_{\text{eff}}(\infty) = 0$. Therefore, the circular orbit is stable whenever it exists.

(b) Let $\ell = \epsilon\, \ell_c$. The effective potential is then

$$U_{\text{eff}}(r) = U_0\, f(r/b) \ ,$$

where the dimensionless effective potential is

$$f(s) = \frac{\epsilon^2}{s^2} - \ln(1 + s^{-2}) \ .$$

The phase curves are plotted in Fig. 2.

(c) The energy is

$$\begin{aligned} E &= \tfrac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\text{eff}}(r) \ , \end{aligned}$$

where we've used $\dot{r} = \dot{\phi}\, r'$ along with $\ell = \mu r^2 \dot{\phi}$. Writing $r = r_0 + \eta$ and differentiating E with respect to ϕ, we find

$$\eta'' = -\beta^2 \eta \qquad , \qquad \beta^2 = \frac{\mu r_0^4}{\ell^2}\, U''_{\text{eff}}(r_0) \ .$$

For our potential, we have

$$\boxed{\beta^2 = 2 - \frac{\ell^2}{\mu b^2 U_0} = 2\left(1 - \frac{\ell^2}{\ell_c^2}\right)}$$

The solution is

$$\boxed{\eta(\phi) = A\, \cos(\beta\phi + \delta)} \tag{2}$$

where A and δ are constants.

(d) The change of periapsis per cycle is

$$\boxed{\Delta\phi = 2\pi\big(\beta^{-1} - 1\big)}$$

If $\beta > 1$ then $\Delta\phi < 0$ and periapsis *advances* each cycle (*i.e.* it comes sooner with every cycle). If $\beta < 1$ then $\Delta\phi > 0$ and periapsis *recedes*. For $\beta = 1$, which means $\ell = \sqrt{\mu b^2 U_0}$, there is no precession and $\Delta\phi = 0$.

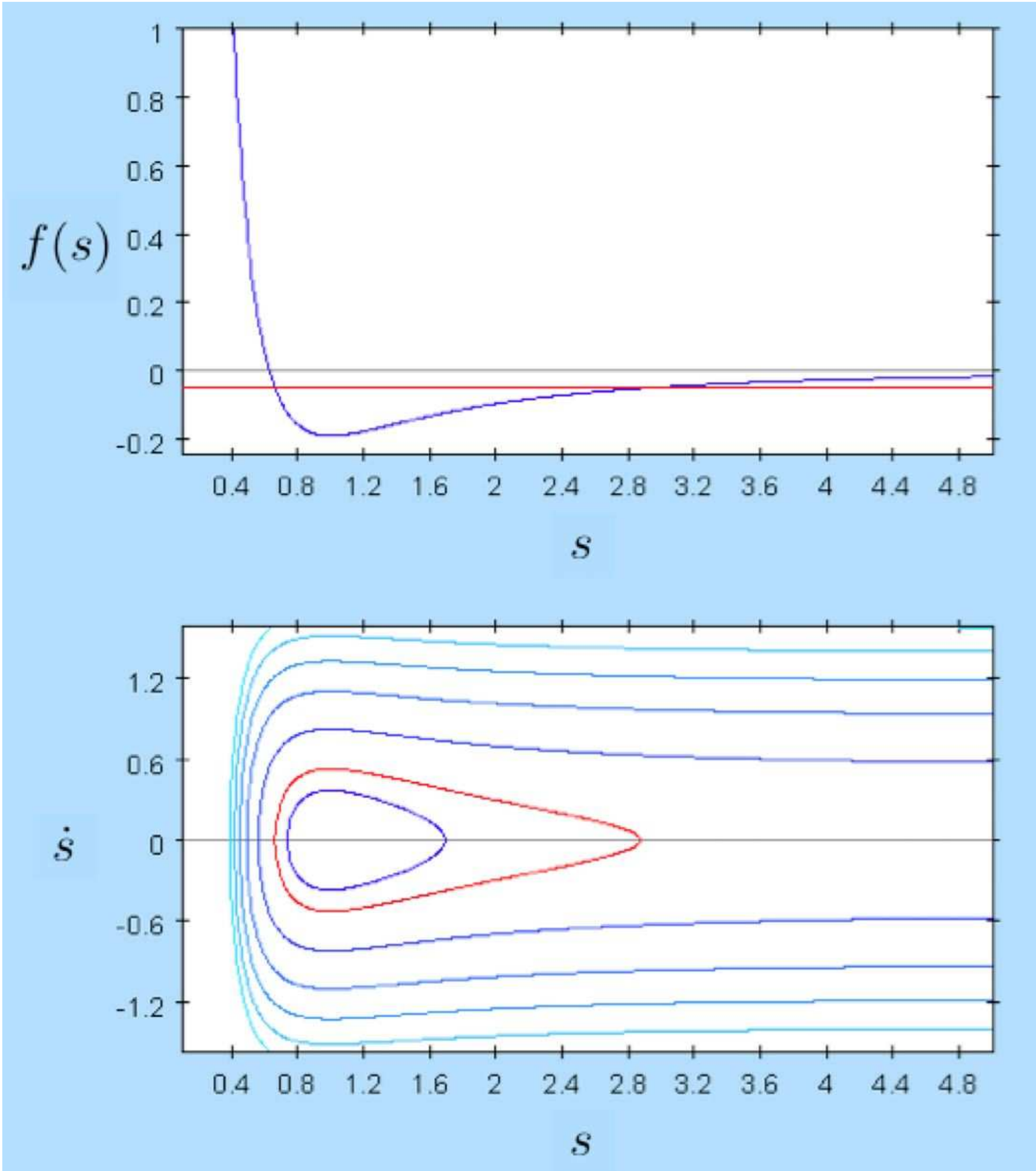

Figure 2: Phase curves for the scaled effective potential $f(s) = \epsilon\, s^{-2} - \ln(1 + s^{-2})$, with $\epsilon = \frac{1}{\sqrt{2}}$. Here, $\epsilon = \ell/\ell_{\rm c}$. The dimensionless time variable is $\tau = t \cdot \sqrt{U_0/mb^2}$.

[3] A particle of charge e moves in three dimensions in the presence of a uniform magnetic field $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$ and a uniform electric field $\boldsymbol{E} = E_0\,\hat{\boldsymbol{x}}$. The potential energy is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = -e\,E_0\,x - \frac{e}{c}\,B_0\,x\,\dot{y}\ ,$$

where we have chosen the gauge $\boldsymbol{A} = B_0\,x\,\hat{\boldsymbol{y}}$.

(a) Find the canonical momenta p_x, p_y, and p_z.
`[7 points]`

(b) Identify all conserved quantities.
`[8 points]`

(c) Find a complete, general solution for the motion of the system $\{x(t), y(t), x(t)\}$.
`[10 points]`

Solution

(a) The Lagrangian is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}\,B_0\,x\,\dot{y} + e\,E_0\,x\ .$$

The canonical momenta are

$$\boxed{p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}}$$

$$\boxed{p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{e}{c}\,B_0\,x}$$

$$\boxed{p_x = \frac{\partial L}{\partial \dot{z}} = m\dot{z}}$$

(b) There are three conserved quantities. First is the momentum p_y, since $F_y = \frac{\partial L}{\partial y} = 0$. Second is the momentum p_z, since $F_y = \frac{\partial L}{\partial z} = 0$. The third conserved quantity is the Hamiltonian, since $\frac{\partial L}{\partial t} = 0$. We have

$$H = p_x\,\dot{x} + p_y\,\dot{y} + p_z\,\dot{z} - L$$

$$\Rightarrow \quad \boxed{H = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\,E_0\,x}$$

(c) The equations of motion are

$$\begin{aligned} \ddot{x} - \omega_{\rm c}\,\dot{y} &= \frac{e}{m}\,E_0 \\ \ddot{y} + \omega_{\rm c}\,\dot{x} &= 0 \\ \ddot{z} &= 0\ . \end{aligned}$$

The second equation can be integrated once to yield $\dot{y} = \omega_{\rm c}(x_0 - x)$, where x_0 is a constant. Substituting this into the first equation gives

$$\ddot{x} + \omega_{\rm c}^2\,x = \omega_{\rm c}^2\,x_0 + \frac{e}{m}\,E_0\ .$$

This is the equation of a constantly forced harmonic oscillator. We can therefore write the general solution as

$$\boxed{x(t) = x_0 + \frac{eE_0}{m\omega_{\rm c}^2} + A\,\cos\big(\omega_{\rm c}t + \delta\big)}$$

$$\boxed{y(t) = y_0 - \frac{eE_0}{m\omega_{\rm c}}\,t - A\,\sin\big(\omega_{\rm c}t + \delta\big)}$$

$$\boxed{z(t) = z_0 + \dot{z}_0\,t}$$

Note that there are six constants, $\big\{A,\ \delta,\ x_0,\ y_0,\ z_0,\ \dot{z}_0\big\}$, are are required for the general solution of three coupled second order ODEs.

[4] An $N = 1$ dynamical system obeys the equation

$$\frac{du}{dt} = ru + 2bu^2 - u^3 \ ,$$

where r is a control parameter, and where $b > 0$ is a constant.

(a) Find and classify all bifurcations for this system.
`[7 points]`

(b) Sketch the fixed points u^* *versus* r.
`[6 points]`

Now let $b = 3$. At time $t = 0$, the initial value of u is $u(0) = 1$. The control parameter r is then increased *very slowly* from $r = -20$ to $r = +20$, and then decreased very slowly back down to $r = -20$.

(c) What is the value of u when $r = -5$ on the *increasing* part of the cycle?
`[3 points]`

(d) What is the value of u when $r = +16$ on the *increasing* part of the cycle?
`[3 points]`

(e) What is the value of u when $r = +16$ on the *decreasing* part of the cycle?
`[3 points]`

(f) What is the value of u when $r = -5$ on the *decreasing* part of the cycle?
`[3 points]`

Solution

(a) Setting $\dot{u} = 0$ we obtain

$$(u^2 - 2bu - r)\, u = 0 \ .$$

The roots are

$$u = 0 \quad , \quad u = b \pm \sqrt{b^2 + r} \ .$$

The roots at $u = u_{\pm} = b \pm \sqrt{b^2 + r}$ are only present when $r > -b^2$. At $r = -b^2$ there is a *saddle-node bifurcation*. The fixed point $u = u_{-}$ crosses the fixed point at $u = 0$ at $r = 0$, at which the two fixed points exchange stability. This corresponds to a *transcritical bifurcation*. In Fig. 3 we plot $\dot{u}/b^3$ *versus* u/b for several representative values of r/b^2. Note that, defining $\tilde{u} = u/b$, $\tilde{r} = r/b^2$, and $\tilde{t} = b^2 t$ that our $N = 1$ system may be written

$$\frac{d\tilde{u}}{d\tilde{t}} = \big(\tilde{r} + 2\tilde{u} - \tilde{u}^2\big)\, \tilde{u} \ ,$$

which shows that it is only the dimensionless combination $\tilde{r} = r/b^2$ which enters into the location and classification of the bifurcations.

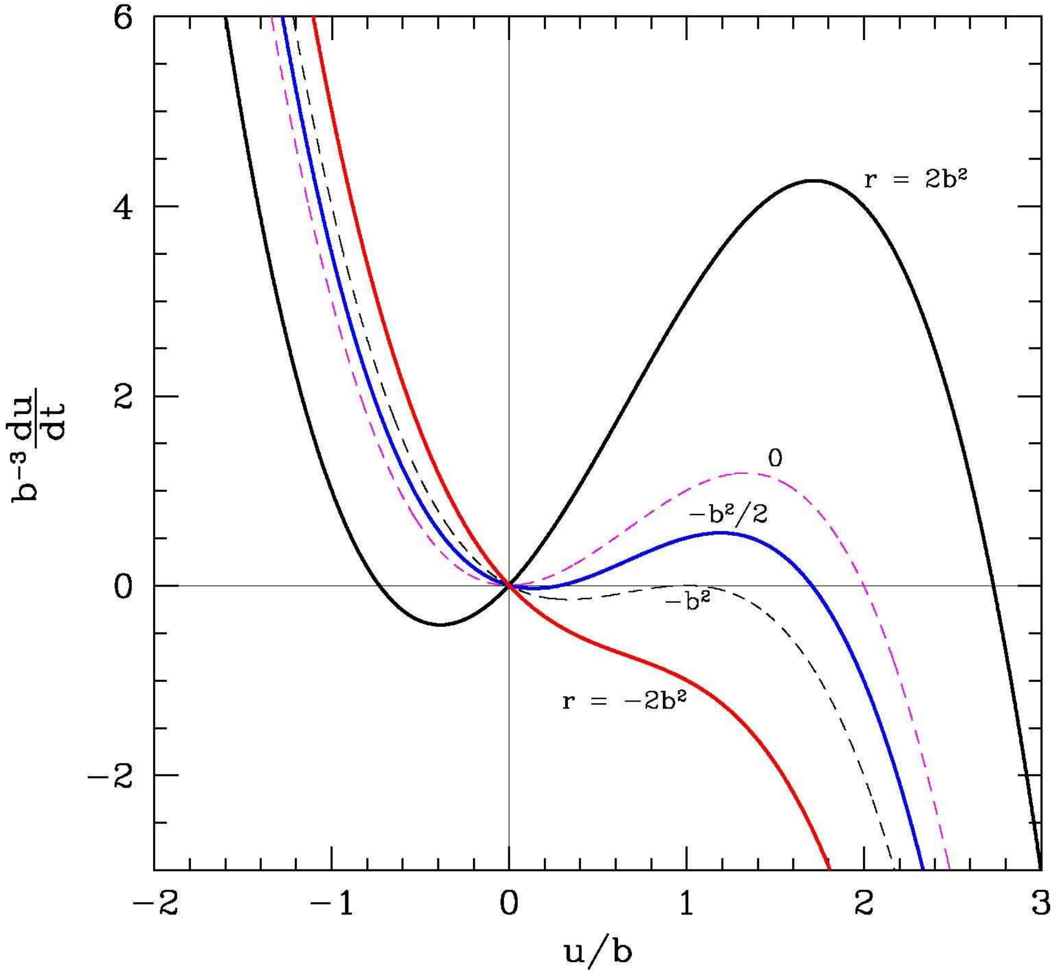

Figure 3: Plot of dimensionless 'velocity' $\dot{u}/b^3$ *versus* dimensionless 'coordinate' u/b for several values of the dimensionless control parameter $\tilde{r} = r/b^2$.

(b) A sketch of the fixed points u^* *versus* r is shown in Fig. 4. Note the two bifurcations at $r = -b^2$ (saddle-node) and $r = 0$ (transcritical).

(c) For $r = -20 < -b^2 = -9$, the initial condition $u(0) = 1$ flows directly toward the stable fixed point at $u = 0$. Since the approach to the FP is asymptotic, u remains slightly positive even after a long time. When $r = -5$, the FP at $u = 0$ is still stable. *Answer:* $\underline{u = 0}$.

(d) As soon as r becomes positive, the FP at $u^* = 0$ becomes unstable, and u flows to the upper branch u_+. When $r = 16$, we have $u = 3 + \sqrt{3^2 + 16} = 8$. *Answer:* $\underline{u = 8}$.

(e) Coming back down from larger r, the upper FP branch remains stable, thus, $u = 8$ at $r = 16$ on the way down as well. *Answer:* $\underline{u = 8}$.

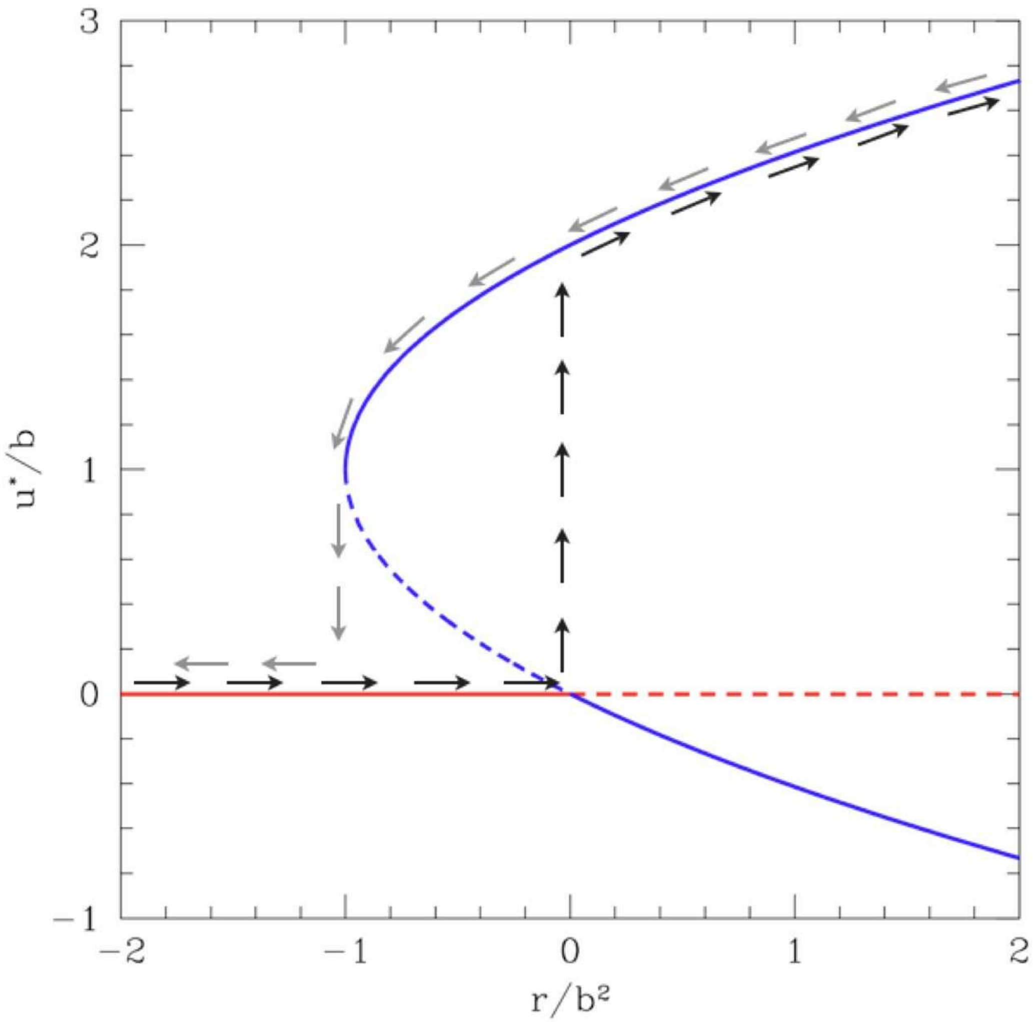

Figure 4: Fixed points and their stability *versus* control parameter for the $N = 1$ system $\dot{u} = ru + 2bu^2 - u^3$. Solid lines indicate stable fixed points; dashed lines indicate unstable fixed points. There is a saddle-node bifurcation at $r = -b^2$ and a transcritical bifurcation at $r = 0$. The hysteresis loop in the upper half plane $u > 0$ is shown. For $u < 0$ variations of the control parameter r are reversible and there is no hysteresis.

(f) Now when r first becomes negative on the way down, the upper branch u_+ remains stable. Indeed it remains stable all the way down to $r = -b^2$, the location of the saddle-node bifurcation, at which point the solution $u = u_+$ simply vanishes and the flow is toward $u = 0$ again. Thus, for $r = -5$ on the way down, the system remains on the upper branch, in which case $u = 3 + \sqrt{3^2 - 5} = 5$. *Answer:* $\underline{u = 5}$.