PHYSICS 110A : CLASSICAL MECHANICS
FINAL EXAM

[1] Two blocks and three springs are configured as in Fig. 1. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

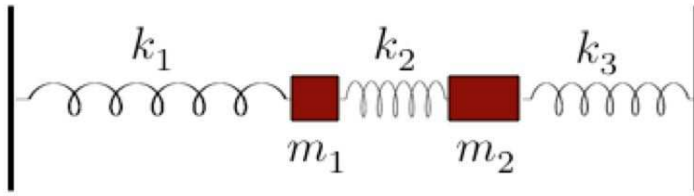

Figure 1: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.
`[5 points]`

(b) Find the T and V matrices.
`[5 points]`

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.
`[5 points]`

(d) Find the normal modes of oscillation.
`[5 points]`

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.
`[5 points]`

[2] Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \, \ln\left(\frac{r^2}{r^2 + b^2}\right) ,$$

where b is a constant with dimensions of length.

(a) For what values of the relative angular momentum ℓ does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?
`[7 points]`

(c) For the case where a circular orbit exists, sketch the phase curves for the radial motion in the $(r, \dot{r})$ half-plane. Identify the energy ranges for bound and unbound orbits.
`[5 points]`

(c) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $|\eta| \ll r_0$. Find the equation for the shape $\eta(\phi)$ of the perturbation.
`[8 points]`

(d) What is the angle $\Delta\phi$ through which periapsis changes each cycle? For which value(s) of ℓ does the perturbed orbit not precess?
`[5 points]`

[3] A particle of charge e moves in three dimensions in the presence of a uniform magnetic field $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$ and a uniform electric field $\boldsymbol{E} = E_0\,\hat{\boldsymbol{x}}$. The potential energy is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = -e\,E_0\,x - \frac{e}{c}\,B_0\,x\,\dot{y}\ ,$$

where we have chosen the gauge $\boldsymbol{A} = B_0\,x\,\hat{\boldsymbol{y}}$.

(a) Find the canonical momenta p_x, p_y, and p_z.
`[7 points]`

(b) Identify all conserved quantities.
`[8 points]`

(c) Find a complete, general solution for the motion of the system $\big\{x(t), y(t), x(t)\big\}$.
`[10 points]`

[4] An $N = 1$ dynamical system obeys the equation

$$\frac{du}{dt} = ru + 2bu^2 - u^3 \ ,$$

where r is a control parameter, and where $b > 0$ is a constant.

(a) Find and classify all bifurcations for this system.
`[7 points]`

(b) Sketch the fixed points u^* *versus* r.
`[6 points]`

Now let $b = 3$. At time $t = 0$, the initial value of u is $u(0) = 1$. The control parameter r is then increased *very slowly* from $r = -20$ to $r = +20$, and then decreased very slowly back down to $r = -20$.

(c) What is the value of u when $r = -5$ on the *increasing* part of the cycle?
`[3 points]`

(d) What is the value of u when $r = +16$ on the *increasing* part of the cycle?
`[3 points]`

(e) What is the value of u when $r = +16$ on the *decreasing* part of the cycle?
`[3 points]`

(f) What is the value of u when $r = -5$ on the *decreasing* part of the cycle?
`[3 points]`