PHYSICS 110A : CLASSICAL MECHANICS
DISCUSSION #2 PROBLEMS

[**1**] Solve the equation

$$L_t\, x \equiv \dddot{x} + (a+b+c)\,\ddot{x} + (ab+ac+bc)\,\dot{x} + abc\,x = f_0 \cos(\Omega t)\ . \tag{1}$$

Solution – The key to solving this was the hint that the differential operator L_t could be written as

$$\begin{aligned} L_t &= \frac{d^3}{dt^3} + (a+b+c)\,\frac{d^2}{dt^2} + (ab+ac+bc)\,\frac{d}{dt} + abc \\ &= \Big(\frac{d}{dt}+a\Big)\Big(\frac{d}{dt}+b\Big)\Big(\frac{d}{dt}+c\Big)\ , \end{aligned} \tag{2}$$

which says that the third order differential operator appearing in the ODE is in fact a product of first order differential operators. Since

$$\frac{dx}{dt} + \alpha x = 0 \quad \Longrightarrow \quad x(t) = A\, e^{-\alpha x}\ , \tag{3}$$

we see that the homogeneous solution takes the form

$$x_{\rm h}(t) = A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct}\ , \tag{4}$$

where A, B, and C are constants.

To find the inhomogeneous solution, we solve $L_t\, x = f_0\, e^{-i\Omega t}$ and take the real part. Writing $x(t) = x_0\, e^{-i\Omega t}$, we have

$$L_t\, x_0\, e^{-i\Omega t} = (a - i\Omega)\,(b - i\Omega)\,(c - i\Omega)\, x_0\, e^{-i\Omega t} \tag{5}$$

and thus

$$x_0 = \frac{f_0\, e^{-i\Omega t}}{(a-i\Omega)(b-i\Omega)(c-i\Omega)} \equiv A(\Omega)\, e^{i\delta}\, f_0\, e^{-i\Omega t}\ ,$$

where

$$A(\Omega) = \Big[(a^2+\Omega^2)\,(b^2+\Omega^2)\,(c^2+\Omega^2)\Big]^{-1/2} \tag{6}$$

$$\delta(\Omega) = \tan^{-1}\Big(\frac{\Omega}{a}\Big) + \tan^{-1}\Big(\frac{\Omega}{b}\Big) + \tan^{-1}\Big(\frac{\Omega}{c}\Big)\ . \tag{7}$$

Thus, the most general solution to $L_t\, x(t) = f_0 \cos(\Omega t)$ is

$$x(t) = A(\Omega)\, f_0\, \cos\big(\Omega t - \delta(\Omega)\big) + A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct}\ . \tag{8}$$

Note that the phase shift increases monotonically from $\delta(0) = 0$ to $\delta(\infty) = \frac{3}{2}\pi$.

[2] Consider the potential

$$U(x) = U_0\,(x^2 - a^2)\,(x^2 - 4a^2)\,(x^2 - 9a^2)\ . \tag{9}$$

Sketch $U(x)$ and the phase curves.

Solution – Clearly $U(x \to \pm\infty) = \infty$, and $U(x)$ has zeros at $x = \pm a$, $x = \pm 2a$, and $x = \pm 3a$. Setting $U'(x) = 0$ we obtain $x = 0$ and also a quadratic equation in x^2, with roots at $x^2 = 7a^2$ and $x^2 = \frac{7}{3}a^2$. Plugging in, we find the three local minima, at $x = \pm\sqrt{7}\,a$ and $x = 0$ are all degenerate, with $U = -36\,U_0\,a^6$, and the two maxima at $x = \pm\sqrt{\frac{7}{3}}\,a$ have $U = \frac{400}{27}\,U_0\,a^6$. This is a nice problem for Ben Schmidel's phase plotter.

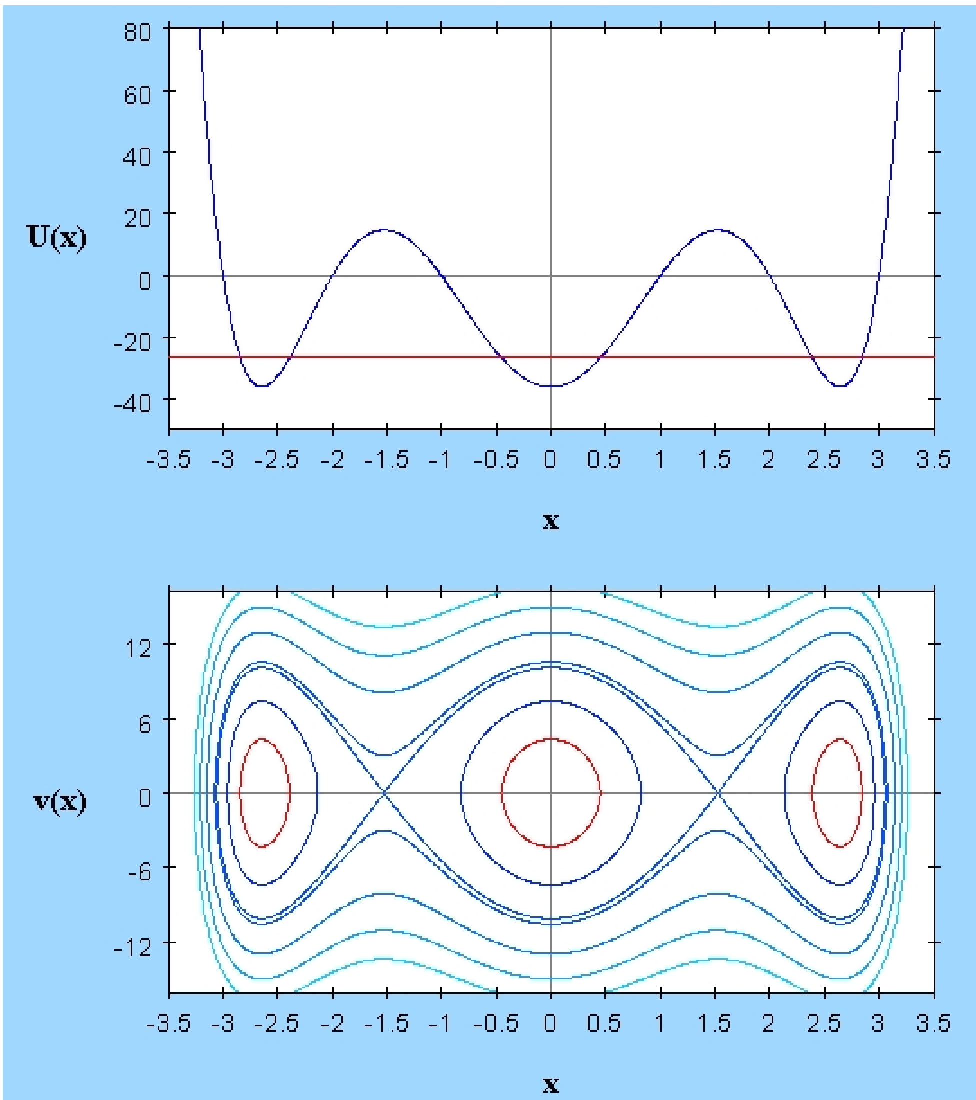

Figure 1: $U(x) = (x^2 - 1)\,(x^2 - 4)\,(x^2 - 9)$ and associated phase curves.

[3] Consider the van der Pol oscillator,

$$\ddot{x} + 2\mu\,(x^2 - 1)\,\dot{x} + x = 0 \ . \tag{10}$$

Find and classify the fixed point(s), find the nullclines, sketch the phase flow, and argue that a stable limit cycle exists.

Solution – With $v = \dot{x}$, we have

$$\dot{x} = v \quad , \quad \dot{v} = -x + \mu(1 - x^2)\,v \ . \tag{11}$$

Since both $\dot{x} = 0$ and $\dot{v} = 0$ at a fixed point, we find a unique fixed point at $(x, v) = (0, 0)$. Linearizing about the fixed point, we write $x = 0 + \delta x$, $v = 0 + \delta v$, with

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}}^{M} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} \ . \tag{12}$$

The matrix M has trace $T = \mu$ and determinant $D = +1$. Thus, according to the fixed point classification scheme derived in class and in the notes, the fixed point $(0, 0)$ is a stable node if $\mu > 2$ and a stable spiral if $\mu < 2$.

The nullclines are curves along which $\dot{x} = 0$ or $\dot{v} = 0$. The equation of the x nullcline is $v = 0$, *i.e.* the x-axis. Along the x-axis, then, the flow must be purely up or down, with no

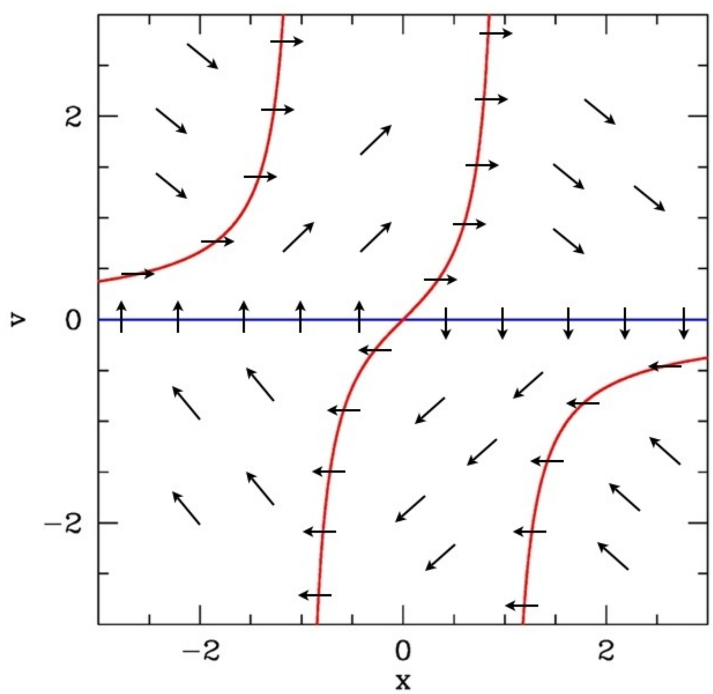

Figure 2: Sketch of phase flow for the van der Pol system. Only the generai direction of the flow is shown. Blue line: x nullcline; red line: v nullcline.

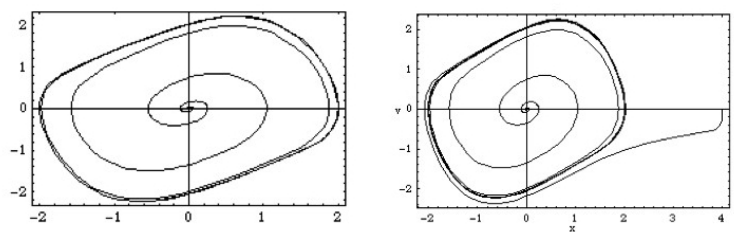

Figure 3: Evolution of the van der Pol equation for $\mu = \frac{1}{2}$, starting from two initial conditions. The flow spirals toward the stable limit cycle.

x component. The equation of the v nullcline is

$$v = \frac{1}{\mu} \frac{x}{1 - x^2} \ . \tag{13}$$

The nullclines and the flow are sketched in Fig. 2. Note that the x-component of the phase velocity $\dot{\boldsymbol{\varphi}}$ changes sign across an x-nullcline, and the v-componend of $\dot{\boldsymbol{\varphi}}$ changes sign across a v-nullcline.

The limit cycle is shown in Figs. 3 and 4.

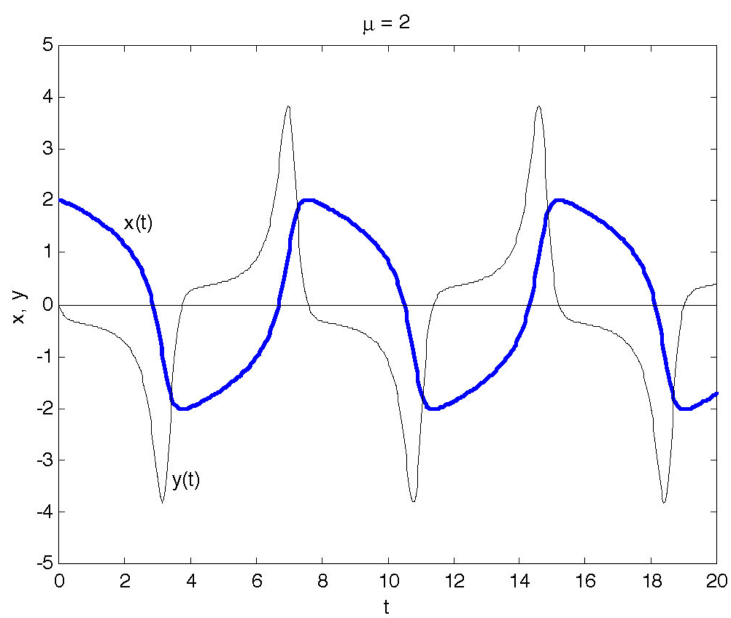

Figure 4: $x(t)$ and $v(t)$ ($y(t)$ in this plot) for the van der Pol system, with $\mu = 2$.

[**4**] Consider the following circuit and construct a mechanical analog.

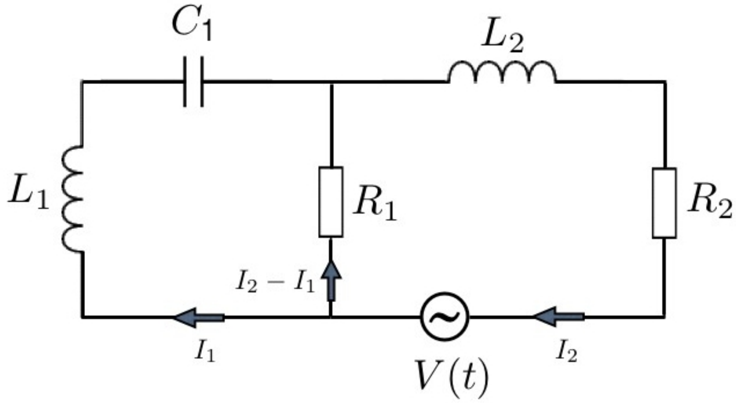

Figure 5: A driven L-C-R circuit, with $V(t) = V_0 \cos(\omega t)$.

Solution – We invoke Kirchoff's laws around the left and right loops:

$$L_1 \dot{I}_1 + \frac{Q_1}{C_1} + R_1 (I_1 - I_2) = 0 \tag{14}$$

$$L_2 \dot{I}_2 + R_2 I_2 + R_1 (I_2 - I_1) = V(t) \ . \tag{15}$$

Let $Q_1(t)$ be the charge on the left plate of capacitor C_1, and define

$$Q_2(t) = \int_0^t dt' \, I_2(t') \ . \tag{16}$$

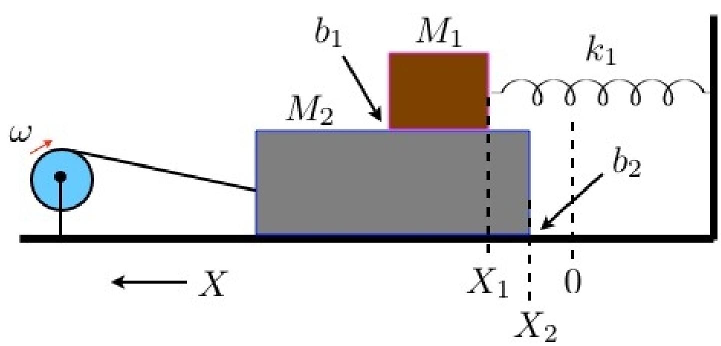

Figure 6: The equivalent mechanical circuit.

Then Kirchoff's laws may be written

$$\ddot{Q}_1 + \frac{R_1}{L_1}\,(\dot{Q}_1 - \dot{Q}_2) + \frac{1}{L_1 C_1}\,Q_1 = 0 \tag{17}$$

$$\ddot{Q}_2 + \frac{R_2}{L_2}\,\dot{Q}_2 + \frac{R_1}{L_2}\,(\dot{Q}_2 - \dot{Q}_1) = \frac{V(t)}{L_2}\ . \tag{18}$$

Now consider the mechanical system in Fig. 6. The blocks have masses M_1 and M_2. The friction coefficient between blocks 1 and 2 is b_1, and the friction coefficient between block 2 and the floor is b_2. There is a spring of spring constant k_1 which connects block 1 to the wall. Finally, block 2 is driven by a periodic acceleration $f_0\,\cos(\omega t)$. We now identify

$$X_1 \leftrightarrow Q_1 \quad , \quad X_2 \leftrightarrow Q_2 \quad , \quad b_1 \leftrightarrow \frac{R_1}{L_1} \quad , \quad b_2 \leftrightarrow \frac{R_2}{L_2} \quad , \quad k_1 \leftrightarrow \frac{1}{L_1 C_1}\ , \tag{19}$$

as well as $f(t) \leftrightarrow V(t)/L_2$.