PHYSICS 4E FINAL EXAM SPRING QUARTER 2009
PROF. HIRSCH June 12

Formulas:

Relativistic energy - momentum relation $E = \sqrt{m^2c^4 + p^2c^2}$; $c = 3\times10^8 m/s$

Electron rest mass : $m_e = 0.511\ MeV/c^2$;Proton : $m_p = 938.26\ MeV/c^2$;Neutron : $m_n = 939.55\ MeV/c^2$

Planck's law : $u(\lambda) = n(\lambda)\bar{E}(\lambda)$; $n(\lambda) = \frac{8\pi}{\lambda^4}$; $\bar{E}(\lambda) = \frac{hc}{\lambda}\frac{1}{e^{hc/\lambda k_BT} - 1}$

Energy in a mode/oscillator : $E_f = nhf$; probability $P(E) \propto e^{-E/k_BT}$

Stefan's law : $R = \sigma T^4$; $\sigma = 5.67\times10^{-8} W/m^2K^4$; $R = cU/4$, $U = \int_0^\infty u(\lambda)d\lambda$

Wien's displacement law : $\lambda_m T = \frac{hc}{4.96k_B}$

Photons : $E = pc$; $E = hf$; $p = h/\lambda$; $f = c/\lambda$

Photoelectric effect : $eV_0 = (\frac{1}{2}mv^2)_{\max} = hf - \phi$, $\phi \equiv$ work function

Compton scattering : $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$

Rutherford scattering: $b = \frac{kq_\alpha Q}{m_\alpha v^2}\cot(\theta/2)$; $\Delta N \propto \frac{1}{\sin^4(\theta/2)}$

Constants : $hc = 12{,}400\ eV\,A$; $\hbar c = 1{,}973\ eV\,A$; $k_B = 1/11{,}600\ \ eV/K$; $ke^2 = 14.4 eVA$

Electrostatics : $F = \frac{kq_1q_2}{r^2}$ (force) ; $U = q_0V$ (potential energy) ; $V = \frac{kq}{r}$ (potential)

Hydrogen spectrum : $\frac{1}{\lambda} = R(\frac{1}{m^2} - \frac{1}{n^2})$; $R = 1.097\times10^7\ m^{-1} = \frac{1}{911.3A}$

Bohr atom : $E_n = -\frac{ke^2Z}{2r_n} = -\frac{Z^2E_0}{n^2}$; $E_0 = \frac{ke^2}{2a_0} = \frac{mk^2e^4}{2\hbar^2} = 13.6eV$; $E_n = E_{kin} + E_{pot}, E_{kin} = -E_{pot}/2 = -E_n$

$hf = E_i - E_f$; $r_n = r_0n^2$; $r_0 = \frac{a_0}{Z}$; $a_0 = \frac{\hbar^2}{mke^2} = 0.529A$; $L = mvr = n\hbar$ angular momentum

X - ray spectra : $f^{1/2} = A_n(Z - b)$; K : $b = 1$, L : $b = 7.4$

de Broglie : $\lambda = \frac{h}{p}$; $f = \frac{E}{h}$; $\omega = 2\pi f$; $k = \frac{2\pi}{\lambda}$; $E = \hbar\omega$; $p = \hbar k$; $E = \frac{p^2}{2m}$

group and phase velocity : $v_g = \frac{d\omega}{dk}$; $v_p = \frac{\omega}{k}$; Heisenberg : $\Delta x\Delta p \sim \hbar$; $\Delta t\Delta E \sim \hbar$

Wave function $\Psi(x,t) = |\Psi(x,t)|e^{i\theta(x,t)}$; $P(x,t)\,dx = |\Psi(x,t)|^2\,dx$ = probability

Schrodinger equation : $-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi(x,t) = i\hbar\frac{\partial\Psi}{\partial t}$; $\Psi(x,t) = \psi(x)e^{-i\frac{E}{\hbar}t}$

Time – independent Schrodinger equation : $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x) = E\psi(x)$; $\int_{-\infty}^{\infty} dx\ \psi^*\psi = 1$

∞ square well : $\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$; $E_n = \frac{\pi^2\hbar^2n^2}{2mL^2}$; $x_{op} = x$, $p_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$; $\langle A\rangle = \int_{-\infty}^{\infty} dx\psi^* A_{op}\psi$

Eigenvalues and eigenfunctions: $A_{op}\ \Psi = a\ \Psi$ (a is a constant) ; uncertainty: $\Delta A = \sqrt{<A^2> - <A>^2}$

Harmonic oscillator: $\Psi_n(x) = C_n H_n(x) e^{-\frac{m\omega}{2\hbar}x^2}$; $E_n = (n+\frac{1}{2})\hbar\omega$; $E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$; $\Delta n = \pm 1$

Step potential: $R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$, $T = 1 - R$; $k = \sqrt{\frac{2m}{\hbar^2}(E - V)}$

Tunneling: $\psi(x) \sim e^{-\alpha x}$; $T \sim e^{-2\alpha\Delta x}$; $T \sim e^{-2\int_a^b \alpha(x)dx}$; $\alpha(x) = \sqrt{\frac{2m[V(x) - E]}{\hbar^2}}$

3D square well: $\Psi(x,y,z) = \Psi_1(x)\Psi_2(y)\Psi_3(z)$; $E = \frac{\pi^2\hbar^2}{2m}(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2})$

Spherically symmetric potential: $\Psi_{n,\ell,m}(r,\theta,\phi) = R_{n\ell}(r) Y_{\ell m}(\theta,\phi)$; $Y_{\ell m}(\theta,\phi) = f_{lm}(\theta) e^{im\phi}$

Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$; $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$; $L^2 Y_{\ell m} = \ell(\ell+1)\hbar^2 Y_{\ell m}$; $L_z = m\hbar$

Radial probability density: $P(r) = r^2 |R_{n,\ell}(r)|^2$; Energy: $E_n = -13.6eV\frac{Z^2}{n^2}$

Ground state of hydrogen and hydrogen-like ions: $\Psi_{1,0,0} = \frac{1}{\pi^{1/2}}(\frac{Z}{a_0})^{3/2} e^{-Zr/a_0}$

Orbital magnetic moment: $\vec{\mu} = \frac{-e}{2m_e}\vec{L}$; $\mu_z = -\mu_B m_l$; $\mu_B = \frac{e\hbar}{2m_e} = 5.79\times 10^{-5} eV/T$

Spin 1/2: $s = \frac{1}{2}$, $|S| = \sqrt{s(s+1)}\hbar$; $S_z = m_s\hbar$; $m_s = \pm 1/2$; $\vec{\mu}_s = \frac{-e}{2m_e} g\vec{S}$

Total angular momentum: $\vec{J} = \vec{L} + \vec{S}$; $|J| = \sqrt{j(j+1)}\hbar$; $|l - s| \le j \le l + s$; $-j \le m_j \le j$

Orbital+spin mag moment: $\vec{\mu} = \frac{-e}{2m}(\vec{L} + g\vec{S})$; Energy in mag. field: $U = -\vec{\mu}\cdot\vec{B}$

Two particles : $\Psi(x_1,x_2) = +/- \Psi(x_2,x_1)$; symmetric/antisymmetric

Screening in multielectron atoms: $Z \to Z_{eff}$, $1 < Z_{eff} < Z$

Orbital ordering:

1s < 2s < 2p < 3s < 3p < 4s < 3d < 4p < 5s < 4d < 5p < 6s < 4f < 5d < 6p < 7s < 6d ~ 5f

$I_n(\lambda) = \int_0^\infty dx\, x^n e^{-\lambda x^2}$; $I_0 = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}$; $I_1 = \frac{1}{2\lambda}$; $I_{n+2} = -\frac{\partial}{\partial\lambda} I_n$

$f_{MB}(E) = Ce^{-E/kT}$; $f_{BE}(E) = \frac{1}{e^\alpha e^{E/kT} - 1}$; $f_{FD}(E) = \frac{1}{e^\alpha e^{E/kT} + 1}$; $n(E) = g(E)f(E)$

$F(v_x,v_y,v_z) = f(v_x)f(v_y)f(v_z)$; $f(v_x) = (\frac{m}{2\pi kT})^{1/2} e^{-mv_x^2/2kT}$ (kinetic theory)

Rotation: $E_R = \frac{L^2}{2I}$, $I = \mu R^2$, vibration: $E_v = \hbar\omega(n + \frac{1}{2})$, $\omega = \sqrt{K/\mu}$, $\mu = m_1 m_2/(m_1 + m_2)$

$g(E) = [2\pi(2m)^{3/2} V/h^3] E^{1/2}$ (translation, per spin) ; Equipartition: $<E> = k_B T/2$ per degree of freedom

Lasers: $B_{12} = B_{21}$, $A_{21}/(B_{21}u(f)) = e^{hf/kT} - 1$; $u(f) = (8\pi hf^3/c^3)/(e^{hf/kT} - 1)$

Justify all your answers to all problems

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Problem 1 (10 pts)
An electron moves in a one-dimensional potential of the form:

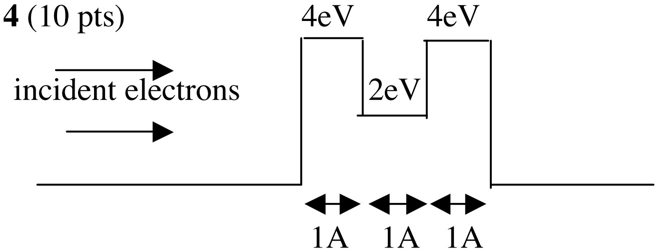
$$V(x) = V_0 \frac{x^4}{x_0^4}$$

with V(x)=400 eV and x_0=1A.
(a) Assume it is known for certain that this electron is within 0.1A of the origin (x=0). Estimate its potential energy, kinetic energy, and total energy. Use Heisenberg's uncertainty principle.
(b) Assume the electron is in the state of minimum energy in this potential. Estimate the range in x that this electron occupies, as well as its potential energy, kinetic energy and total energy.
(c) Explain why the total energy found in (b) is much smaller than the total energy found in (a).
Use $\hbar^2 / m_e = 7.62 \; eV \, A^2$

Problem 2 (10 pts)
An electron in a hydrogen-like ion is in a state with energy -13.6eV and magnitude of angular momentum $3.46\hbar$.
(a) What are the possible values of the ionic charge Z? Justify your answer.
Assume for the next questions that Z is the smallest of the possible values found in (a) and the electron is in the state with energy -13.6eV.
(b) Give the wavelengths of all the photons that this system could emit when the electron makes a transition to a lower energy state, in A (Angstrom).
(c) Give the radius of the orbit of this electron according to the Bohr model, in A.
(d) Find the speed of this electron according to the Bohr model. Give your answer as v/c.

Problem 3 (10 pts)
An electron is in the ground state of an infinite square well of width L=3A.
(a) Find the uncertainty in the momentum of this electron, Δp. Give your answer in units eV/c (in other words, find cΔp in eV). Hint: use that $E=p^2/2m_e$, no integrals needed.
(b) Find the probability that the electron will be found at distance L/6 or less from the left wall. Use $\sin^2 y = (1 - \cos(2y))/2$. Hint: you need to do an integral.
(c) Same as (b) when the electron is in the second excited state (n=3).

Problem 4 (10 pts)



A beam of 20,000 electrons is incident on the barrier shown in the figure. Half of the electrons have energy 1eV and the other half have energy 3eV. Estimate how many electrons of what energy will be found on the other side of the barrier.
Use $\hbar^2 / 2m_e = 3.81 \; eV \, A^2$

Problem 5 (10 pts)
A two-dimensional infinite square well with side lengths $L_1=L_2=6A$ has 5 electrons. Assume electrons don't interact with each other.
(a) Find the ground state energy (in eV) taking into account that electrons are fermions with spin ½.
(b) How many different states are there with the same ground state energy? In other words, find the degeneracy of the ground state.
(c) A magnetic field of magnitude 10T is turned on. Assume there is no orbital angular momentum and consider the interaction of the electrons intrinsic magnetic moment due to spin with the magnetic field. How many different states are there now with the lowest possible energy?
(d) What is the difference in energy (in eV) between a state with the lowest possible energy and one with the next lowest possible energy in the presence of the 10T magnetic field?

Problem 6 (10 pts)
Consider a four-dimensional box of volume L^4, where L is the length of the box.
(a) Generalize the formula for the energy of a non-relativistic particle that you learned in three and lower dimensions to four dimensions.
(b) The number of energy eigenstates with energy less than E can be written as
$N(E) = CE^{\alpha}$
where C and α are constants. Find α.
Hint: the volume of a sphere of radius R in a space of d dimensions is proportional to R^d.
(c) Find the energy dependence of the density of states g(E) in this four-dimensional world. (In our three-dimensional world, $g(E) \propto E^{1/2}$).
(d) What would be the heat capacity of 1 mol of a monatomic gas at constant volume in this four-dimensional world? Justify your answer.

Problem 7 (10 pts)
The n=2, $\ell = 1$ radial wave function for an electron in a hydrogen-like ion with atomic number Z is
$R(r) = Cre^{-2r/a_0}$
with C a constant and a_0 the Bohr radius.
(a) Find the most probable r in terms of a_0.
(b) Give the value of Z for this ion. Justify your answer by comparison with the Bohr atom.
(c) Ignore the spin of the electron. Give the wave function for this electron (except for its normalization constant) as function of r, θ, ϕ in the lowest energy state with n=2, $\ell = 1$ in the presence of a magnetic field B pointing (i) in the +z direction, and (ii) in the -z direction. Give also all its quantum numbers for each of the two cases (i), (ii).
Hint: the θ dependence gives maximum amplitude in the x-y plane.

(see **Problem 8** in next page)

Problem 8 (10 pts)
Consider a system of atoms and radiation in thermal equilibrium. The atoms have ground state energy E_1 and first excited state energy E_2, with $E_2-E_1=1eV$. Assume the ground state is non-degenerate and the first excited state is doubly degenerate.
(a) Find the ratio of the number of atoms with energy E_2 ($\equiv N_2$) to the number of atoms with energy E_1 ($\equiv N_1$) at T=300K.
(b) Find a range of temperature where "population inversion" in thermal equilibrium will occur, i.e. where there are more atoms with energy E_2 than atoms with energy E_1.
(c) Using the dynamical equilibrium condition
$N_1 B_{12} u(f) = N_2 A_{21} + N_2 B_{21} u(f)$
with B_{21}, A_{21}, B_{12}=probability of stimulated emission, spontaneous emission and absorption per atom, and $u(f)$ the energy density of the radiation,
find B_{21}/B_{12} for this system. Hint: the answer is not $B_{21}/B_{12}=1$.

Justify all your answers to all problems