Recap: Linear Perturbation Theory

- Growth of density perturbation with linear scales $\lambda >$ horizon; i.e., $\lambda > l_H$

$$\frac{\delta\rho}{\rho} \propto \begin{cases} a^2; & t < t_{eq} \\ a; & t > t_{eq} \end{cases}$$

- What happens to this mode after it enters the horizon; i.e., at $t >$ $t_{enter}(\lambda)$ where $\lambda = l_H(t_{enter}(\lambda))$?

  Growth suppressed by 2 effects:

  (1) Jeans instability:

  If pressure distribution can build up fast enough before gravitational collapse occurs, the pressure will prevent further growth of $\delta\rho/\rho$ by grav. collapse

$$t_{\substack{\text{pressure}\\ \text{readjustment}}} < t_{\substack{\text{grav.}\\ \text{collapse}}}$$

$$t_{p.r} \simeq \frac{\lambda}{v} \; ; \; v = \text{velocity dispersion}$$

$$t_{grav} \simeq \frac{1}{(G\rho)^{1/2}} = \text{free-fall time}$$

$\therefore$ Stabilization criterion : $\dfrac{\lambda}{v} < \dfrac{1}{(G\rho)^{1/2}}$

So if $\lambda \lesssim \lambda_J = \dfrac{\sqrt{\pi}\, v}{(G\rho)^{1/2}}$, stabilization occurs

Note:

Details of Jeans criterion depend on species involved

(i) <u>One component</u>: If Universe contains only one species, then v and ρ correspond to that species

(ii) <u>Multicomponent</u>:
- $\{v\}$: is velocity dispersion of perturbed component providing pressure (gradient) support.
- $\{\rho\}$: Is density of component dominating gravity that induces collapse.

<u>Baryons</u>: Collision-dominated fluid where $v = c_s$, i.e., the sound speed.

<u>Dark Matter</u>: Here collisions are unimportant. "pressure" support arises from orbit re-adjustment.

(2) <u>Expansion Speed</u>: Growth of perturbations suppressed when:

ⓐ Perturbed species is <u>not</u> species that dominates expansion rate

ⓑ Dominant species is smooth background

ⓒ Example:

Suppose $t_{grav.}$ (for perturbed species say DM) $< t_{pressure}$ - indicating $\lambda > \lambda_J$ and pressure shouldn't prevent collapse. However if we are in radiation-dominated (RD) phase,

Expansion time-scale $\Big\}\; t_{exp} \sim \dfrac{1}{(G\rho_{dominant})^{1/2}} \simeq \dfrac{1}{(G\rho_R)^{1/2}} < t_{grav}$

where t_{grav} was set by DM

Result: $t_{exp} < t_{grav} < \frac{\lambda}{v}$

$$\left(\frac{1}{(G\rho_R)^{1/2}}\right) < \frac{1}{(G\rho_{DM})^{1/2}} < \frac{\lambda}{v}$$

In that case rapid expansion rather than pressure support prevents growth of $\delta\rho/\rho$. So, even though perturbation is "Jeans unstable", collapse does not occur.

Result:

$t < t_{eq}$ (1) In RD phase, rapid expansion prevents growth of all modes with $\lambda < l_H$. In this phase only modes with $\lambda > l_H$ can grow and they grow like $\frac{\delta\rho}{\rho} \propto a^2$

$t > t_{eq}$ (2) In MD phase, all modes with $\lambda > \lambda_J$ grow, even if they are subhorizon, since pressure effects are negligible. In this case $\delta\rho/\rho \propto a$.

Fate of perturbation with with proper wavelength λ

(1) In linear regime $\lambda \propto a$

(2) Perturbation enters horizon in RD phase when $a = a_{enter} = a(t_{enter})$ (set by $\lambda = l_H(t_{enter}(\lambda))$

(3) Consider perturbations in DM (first)

(a) Velocity dispersion $v \simeq c$ at $a < a_{nr}$ ($T > \frac{mc^2}{k}$)

at $a > a_{nr}$ (b) Recall, since $p \sim \frac{1}{a}$, then $v \propto \frac{1}{a}$: $a > a_{nr}$

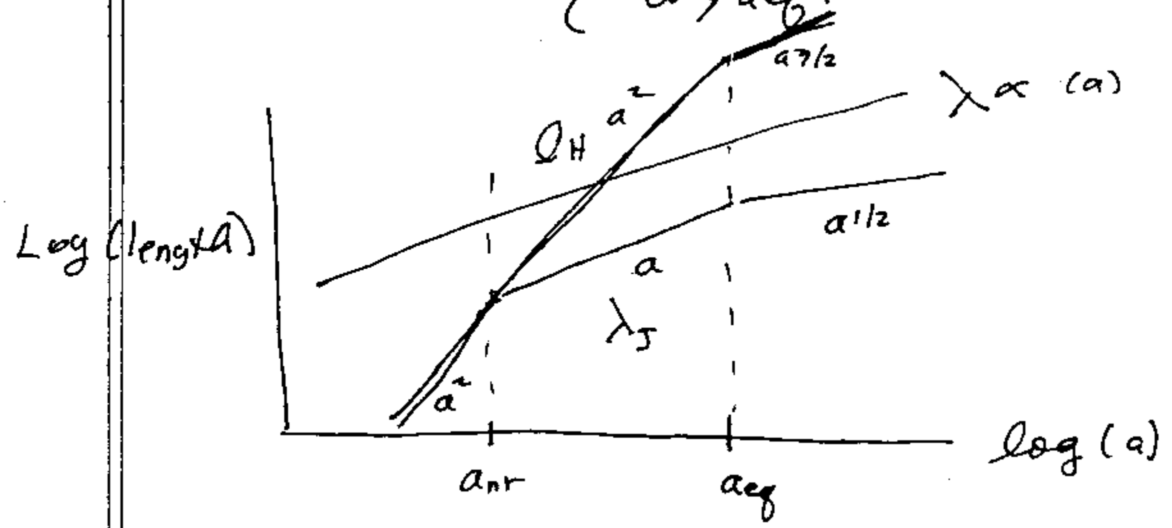
(b) Dominant species:

(4) Jeans Length:

$$\lambda_J \propto \frac{v}{\rho_{DOM}^{1/2}} \propto \begin{cases} a < a_{nr}: & \frac{c}{(1/a^4)^{1/2}} \propto a^2 \\ a_{nr} < a < a_{eq}: & \frac{1/a}{1/a^2} \propto a \\ a > a_{eq}: & \frac{1/a}{(1/a^3)^{1/2}} \propto a^{1/2} \end{cases}$$

(5) Horizon:

$$\lambda_H \propto ct \propto \begin{cases} a < a_{eq}: & \propto t_d\, a^2 \\ a > a_{eq}: & \propto t_d\, a^{3/2} \end{cases}$$



Three Stages of Mode Evolution

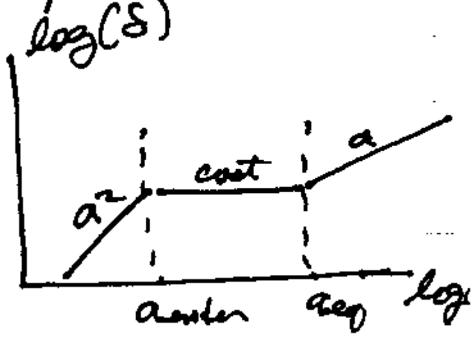
(A) $\boxed{a < a_{enter}:}$
Since $\lambda > \ell_H$: $\delta\rho/\rho \propto a^2$

(B) $\boxed{a_{enter} < a < a_{eq}:}$
In this case, $\lambda < \ell_H$ and $\lambda > \lambda_J$. Pressure support cannot stop collapse. But since $\rho_R > \rho_{DM}$, $t_{exp} < t_{collapse}$: Expansion stops collapse. $\delta\rho \propto \ln(a)$

(C) $\boxed{a > a_{eq}:}$

Since wavelength is inside horizon and $\lambda \gg \lambda_J$ and $\rho_{rem} = \rho_{DM}$; i.e., dominant species for both perturbation and background. Neither pressure support or rapid expansion can suppress growth of density perturbations. Therefore

$$\boxed{\frac{\delta\rho}{\rho} \propto a}$$

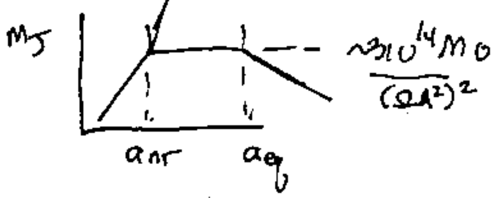


Mass Scales

Conventional to discuss things in terms of the Jeans mass:

$$M_J = \frac{4\pi}{3}\rho\left(\frac{\lambda_J}{2}\right)^3 \text{ where } \rho = \rho_{\text{perturbation}}$$

$$\therefore M_J \propto \underset{\substack{\text{mass in} \\ \text{perturbation}}}{\rho_{DM}} \cdot \lambda_J^3 \propto \begin{cases} a^{-4} \cdot a^6 \propto a^2 & a < a_{nr} \\ a^{-3} \cdot a^3 \propto \text{const.} & a_{nr} < a < a_{eq} \\ a^{-3} \cdot a^{-3/2} \propto a^{-3/2} & a > a_{eq} \end{cases}$$

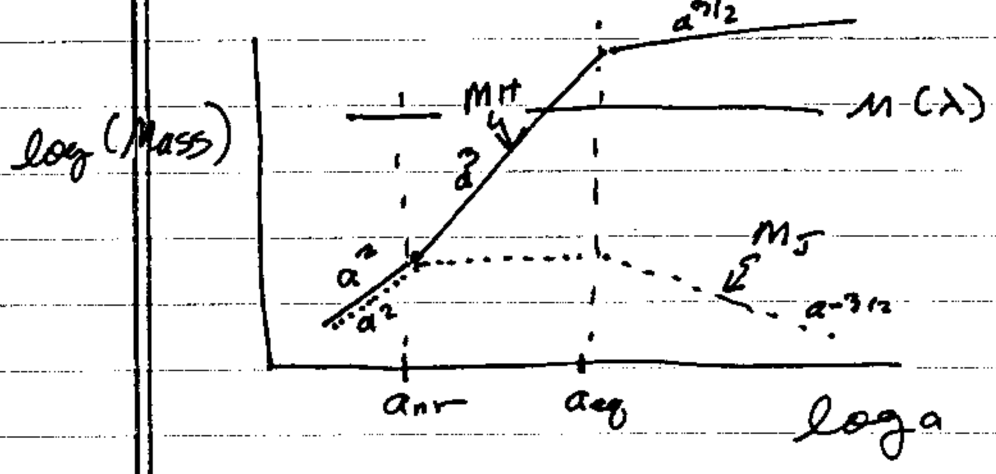


Horizon Mass

$$M_H = \frac{4\pi}{3}\rho_{DM}\left(\frac{l_H}{2}\right)^3$$

$$M_H \propto \begin{cases} a^{-4} \cdot a^6 \propto a^2 & a < a_{nr} \\ a^{-3}(a^2)^3 \propto a^3 & a_{nr} < a < a_{eq} \\ a^{-3}(a^{3/2})^3 \propto a^{3/2} & a > a_{eq} \end{cases}$$

$M_{\text{perturbation}} = M(\lambda) = \text{const.}$



Numerics: $M_J(a_{eq}) = \frac{4\pi}{3}\rho_m(a_{eq})\left(\frac{\lambda_J(a_{eq})}{2}\right)^3 = \frac{4\pi}{3\cdot 8}\cdot\rho_{crit}\Omega_m(1+z_{eq})^3\cdot\lambda_J^3(a_{eq})$

Baryons:

For $a < a_{dec}$, baryons and photons are tightly coupled.

Let $a_{B\gamma}$ be epoch where $\rho_B = \rho_\gamma$

$\rho_\gamma = (1+z)^4\rho_\gamma(0)$; $\rho_B = (1+z)^3\rho_B(0)$

therefore $(1+z)^4\rho_\gamma(0) = (1+z)^3\rho_B(0)$ condition sets $a_{B\gamma}$ or $z_{B\gamma}$ $\Rightarrow$

$$1+z_{B\gamma} = \frac{\rho_B(0)}{\rho_\gamma(0)} = \frac{\Omega_B(3H_0^2/8\pi G)}{a_R\cdot T_0^4}$$

$$\Rightarrow 1+z_{B\gamma} = 3.9\times10^4(\Omega_b h^2)$$

Since $\Omega_b h^2 = 0.02 \Rightarrow 1+z_{B\gamma} \simeq 800$. But baryons cannot be coupled to photons at $z < z_{decoup}$ because gas has become neutral by then: $z_{B\gamma\,\text{effective}} \simeq z_{dec}$.

• $a < a_{B\gamma} \simeq a_{dec}$: $P_R \gg P_B$, $\rho_R > \rho_B$

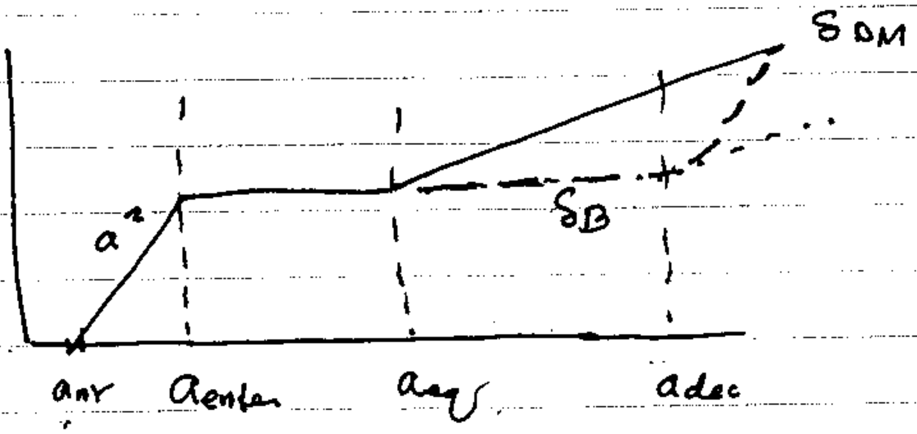
$$v^2 = \frac{\partial P}{\partial\rho} \simeq \frac{P}{\rho} \simeq \frac{P_R + P_B}{\rho_R + \rho_B} \simeq \frac{P_R}{\rho_R} = \frac{1}{3}c^2$$

Since Jeans length of photons $\lambda_J \simeq l_H$, photon fluid contained in perturbation cannot collapse. Since baryons are strongly coupled to photons, $(\delta\rho/\rho)_B \simeq$ const $a_{enter} < a < a_{dec}$.

therefore

$$\delta_B = \left(\frac{\delta\rho}{\rho}\right)_B = \begin{cases} a^2 \; ; & a < a_{enter} \\ \text{const} \; ; & a_{enter} < a < a_{dec} \\ a \; ; & a > a_{decoup} \end{cases}$$

So there is a difference in behaviour between δ_{DM} and δ_B since DM is not coupled to photons



Conclude

(1) Perturbations in DM grow when $a > a_{eq}$. From $a_{eq} \to a_{deco}$, δ_{DM} grows by a factor

$$\frac{a_{dec}}{a_{eq}} \simeq \frac{1+z_{eq}}{1+z_{dec}} \qquad \boxed{\cancel{\frac{a_{eq}}{a_{eq}} \simeq \frac{T_{eq}}{T_{dec}}}}$$

$$\simeq \frac{4\times10^4\,\Omega_m h^2}{1100} \simeq 37\,\Omega_m h^2 \simeq 15$$

(2) Baryon Perturbation does not grow

during this period, i.e., baryons lag behind.

(3) But baryons feel pull of potential DM ~~pot~~ and rapidly fall into potential wells
Baryons catch up

Summary (Fate of Perturbation δ)

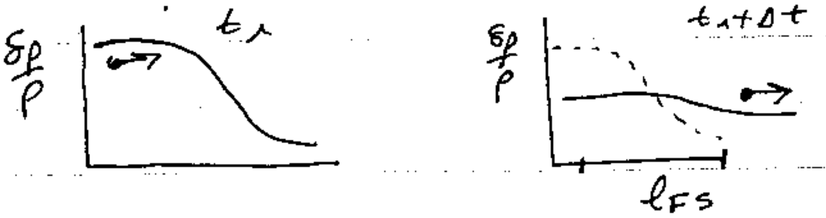
Epoch	Photons (δ_γ)	DM (δ_{DM})	Baryons (δ_B)
$t < t_{enter}$, $\lambda > l_H$	$\propto a^2$	$\propto a^2$	$\propto a^2$
$t_{enter} < t < t_{eq}$, $\lambda < l_H$	oscillates	$\propto \ln a$	oscillates
$t_{eq} < t < t_{dec}$, $\lambda < l_H$	oscillates	$\propto a$	oscillates
$t > t_{dec}$, $\lambda < l_H$	oscillates	$\propto a$	$\propto a$

$\rho_R \sim T^4$

Free-streaming:

On small scales, random motions, i.e., geodesic motions, of DM particles will wipe out density structures, since particles can move from over-dense regions to underdense ones.

Let $l_{FS}(t)$ be proper distance DM particle travels in time t. Then modes with proper wavelength $\lambda(t) < l_{FS}(t)$ will "dissipate" owing to free-streaming.



(1) comoving distance particle travels in $t' \to t'+dt'$

$$dL' = \frac{v\,dt'}{a(t')}, \quad L(t) = \int_0^t \frac{v\,dt'}{a(t')}$$

(2) proper distance: $= a(t)\int_0^t \frac{v(t')}{a(t')}dt'$

$$\boxed{l_{FS}(t) = a(t)\int_0^t \frac{v(t')\,dt'}{a(t')}}$$

Crucial for $a < a_{nr}$ when $v(t') = c$, $a(t) \propto t^{1/2}$

$$l_{FS}(t) = a(t)\,c\int_0^t \frac{dt'}{a(t')} = 2ct$$

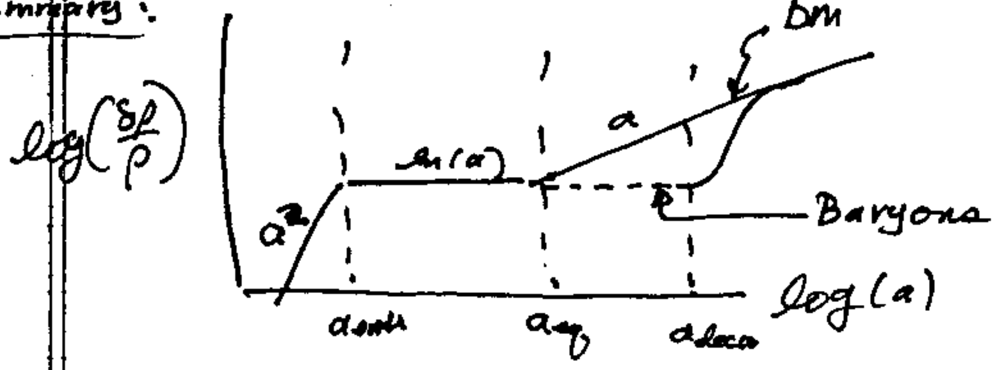
Largest $l_{FS}(t) \simeq l_{FS}(t_{nr}) \simeq 2ct_{nr}$

Today $l_{FS}(t_0) \simeq \frac{a_0}{a_{nr}} \times 2ct_{nr}$

Recap :

Evolution of perturbation with initial $\lambda > l_H$.

(1) Summary :

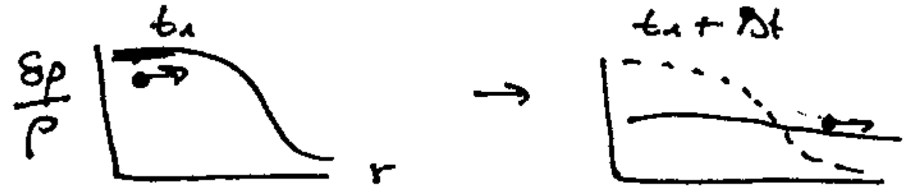


This is a DM perturbation. δ goes like a^2 when $\lambda > l_H$ since

$$\frac{\delta\rho}{\rho} \propto \frac{1}{\rho_0 a_0^2}$$

and ρ dominated by radiation and therefore $\rho_0 \propto a_0^{-4}$.

(2) Free Streaming :



Proper length : $l_{FS}(t) = a(t)\int_0^t \frac{v(t')\,dt'}{a(t')}$

(A) $0 < t < t_{nr}$: $v(t') \simeq c$

$l_{FS}(t) = 2ct$: it's just the horizon scale!

(B) $t_{nr} < t < t_{eq}$: $v(t') = c\,(a_{nr}/a(t'))$

$$l_{FS}(t) = l_{FS}(t_{nr}) + a(t)\int_{t_{nr}}^{t} \frac{dt'}{a(t')} \times \frac{c \cdot a_{nr}}{a(t')}$$

not much bigger than $l_{FS}(t_{nr})$

So the largest ~~distance~~ ~~amount~~ contribution to $l_{FS}(t) \simeq l_{FS}(t_{nr})$ since that is when $v \simeq c$.

Current value

$$\lambda_{FS} \equiv l_{FS}(t_0) \simeq \frac{a_0}{a_{nr}} c t_{nr}$$

• Identify t_{nr}: epoch at which DM particles go non-relativistic: $T_{DM} \simeq \frac{m_x c^2}{3 k_B}$

⋮

$$\lambda_{FS} \simeq .5 (\Omega_{DM} h^2)^{1/3} (1\,\mathrm{keV}/m_x c^2)^{-4/3}\ \mathrm{Mpc}$$

Length-scale below which perturbations are smeared out.

Neutrinos: $\Omega_{DM} h^2 = \Omega_\nu h^2 \simeq \left(\frac{m_\nu}{90\,\mathrm{eV}}\right)$

Neutrinos

$$\Rightarrow \boxed{\lambda_{FS} \simeq 30\,\mathrm{Mpc} \left(\frac{30\,\mathrm{eV}}{m_\nu}\right)}$$

$$M_{FS} \simeq 4 \times 10^{15} \left(\frac{m_\nu}{30\,\mathrm{eV}}\right)^{-2} M_\odot$$

In such a Universe small-scale power wiped out

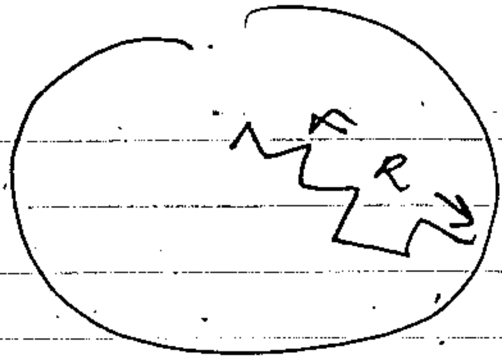
CDM

$$\boxed{\lambda_{FS} \simeq 0.5 \left(\frac{1\,\mathrm{keV}}{m_x}\right)^{4/3}}$$

$$M_{FS} \simeq 6 \times 10^9 M_\odot$$

small-scale power survives

Silk Damping: Occurs in photon-Baryon Plasma



photons random walk and leak out of coupled fluctuation

diffuse out of overdense region

• MFP due to Thomson Scattering $= l_T = \frac{1}{n_e \sigma_{Thom}}$

Random walk:

$$R \simeq \sqrt{N}\, l_T \quad : \; N = \#\text{scatt to reach } R$$

• Time ~~Be~~: Diffusion before $t = t_{dec}$

$$t_{dec} \simeq \frac{N \cdot l_T}{c}$$

$$\Rightarrow \boxed{R = l_T \sqrt{\frac{c\,t_{dec}}{l_T}} \simeq \sqrt{l \cdot c\,t_{dec}}}$$

$$R_{SD} \simeq 3.5\,\text{Mpc}\,(\Omega_m/\Omega_B)^{1/2}(\Omega_m h^2)^{-3/4}$$

$$M_{SD} \simeq 6\times10^{12} M_\odot\,(\Omega/\Omega_B)^{3/2}(\Omega_m h^2)^{-5/2}$$

Processed Final Spectrum

Processed Final Spectrum

We now have essential ingredients to evolve perturbations from some initial time (where their amplitudes, i.e., power spectrum were presumably predicted by inflation) $t_i \ll t_{eq}$ to some final value at $t > t_{dec}$.

DM: Let $\delta_\lambda(t_i)$ = amplitude of DM perturbation corresponding to wavelength at t_i.

- For each λ we have wave no. $k = 2\pi/\lambda$
- mass $M \propto \lambda^3$

Label: We can label perturbations as

$$\delta_M(t),\ \delta_\lambda(t)$$

Henceforth λ will be comoving (value it would have today) ; $\lambda_{phys} = \frac{a(t)}{a(t_0)}\lambda = \frac{\lambda}{1+z}$

Goal: Compute $\delta_\lambda(t)$ or $\delta_M(t)$ at $t \geq t_{dec}$.

① Free Streaming Result: $\lambda < \lambda_{FS}$

Perturbations with $\lambda < \lambda_{FS}$ or $M < M_{FS}$ are wiped out on such scales.

$$\therefore \boxed{\delta_M(t) \approx 0 \quad (M < M_{FS},\ \lambda < \lambda_{FS})}$$

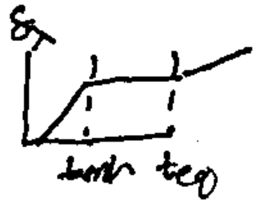
② $\lambda_{FS} < \lambda < \lambda_{eq}$:

These modes enter the horizon in the RD phase ($t \leq t_{eq}$). λ_{eq} defined by the physical scale = horizon scale at $t = t_{eq}$: $(\lambda_{phys})_{eq} = a(t_{eq})\lambda_{eq} = l_H(t$

Growth of δ suppressed by rapid expansion of background.

Therefore

(a) $\delta_\lambda(t_{eq}) \simeq \delta_\lambda(t_{enter})$

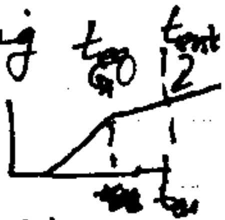


(b) After t_{eq} in MD phase they grow according to:

$$\boxed{\delta_M(t) = \delta_M(t_{enter})\left(\frac{a}{a_{eq}}\right)} \quad \begin{pmatrix} M_{FS} < M < M_{eq} \\ a > a_{eq} \\ \lambda_{FS} < \lambda < \lambda_{eq} \end{pmatrix}$$

(3) $\lambda_{eq} < \lambda <$ ~~[illegible]~~ λ_H ← comoving horizon

Modes with wavelengths between λ_{eq} and Horizon scale at time t. They enter during MD phase and thus grow like $\propto a$



$$\delta_M(t) = \delta_M(t_{enter})\left(\frac{a}{a_{enter}}\right) : M_{eq} < M < M_H$$

Therefore: $\delta_M(t) = \delta_M(t_{enter})\left(\frac{a}{a_{eq}}\right)\left(\frac{a_{eq}}{a_{enter}}\right)$

But time scales fixed by

(i) t_{enter}: $\quad l(t_{enter}) = \lambda\, a(t_{enter}) = l_H(t_{enter})$

Defined by $l_H(t_{enter}) \propto t_{enter}$ (Horizon!)

$\therefore\ t_{enter} \propto \lambda\, t_{enter}^{2/3} \Rightarrow$

$t_{enter} \propto \lambda^3$

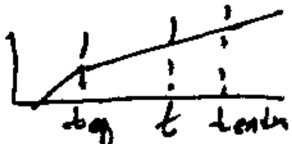
(ii) λ_{eq}.. $\quad l(t_{eq}) = \lambda_{eq}\, a(t_{eq}) = l_H(t_{eq})$

$\therefore\ t_{eq} \propto \lambda_{eq} t_{eq}^{2/3} \Rightarrow$

$t_{eq} \propto \lambda_{eq}^3 \Rightarrow t_{eq}^{2/3} \propto \lambda_{eq}^2$

$$\therefore \left(\frac{a_{eq}}{a_{enter}}\right) = \left(\frac{t_{eq}}{t_{enter}}\right)^{2/3} = \left(\frac{\lambda_{eq}}{\lambda}\right)^2 = \left(\frac{M_{eq}}{M}\right)^{2/3}$$

$$\boxed{\therefore\ \delta_M(t) = \delta_M(t_{enter})\left(\frac{M_{eq}}{M}\right)^{2/3}\left(\frac{a}{a_{eq}}\right) : \begin{matrix} M_{eq} < M < M_H(t) \\ \lambda_{eq} < \lambda < \lambda_H(t) \end{matrix}}$$



(4) $\lambda > \lambda_H$: Modes still outside horizon at time t. Enter horizon at $t_{enter} > t$. Between (t, t_{enter}) they grow by a factor a_{enter}/a. Thus

$$\delta_\lambda(t_{enter}) = \frac{a_{enter}}{a(t)} \delta_\lambda(t)$$

$$\therefore \quad \delta_\lambda(t) = \delta_\lambda(t_{enter}) \frac{a(t)}{a_{enter}} = \delta_\lambda(t_{enter}) \left(\frac{a}{a_{eq}}\right)\left(\frac{a_{eq}}{a_{enter}}\right)$$

From (3) we have

$$\boxed{\delta_\lambda(t) = \delta_\lambda(t_{enter}) \left(\frac{M_{eq}}{M}\right)^{2/3} \left(\frac{a}{a_{eq}}\right) \quad : M > M_H \quad \lambda > \lambda_H}$$

Conclude: (i) Same behavior for $\lambda \geq \lambda_{eq}$

(ii) Summary

$$\delta_M(t) = \begin{cases} 0 & : M < M_{FS} \\ \delta_M(t_{enter})\left(\frac{a}{a_{eq}}\right) & : M_{FS} < M < M_{eq} \\ \delta_M(t_{enter})\left(\frac{M_{eq}}{M}\right)^{2/3}\left(\frac{a}{a_{eq}}\right) & : M > M_{eq} \end{cases}$$

or

$$\delta_\lambda(t) = \begin{cases} 0 & : \lambda < \lambda_{FS} \\ \delta_\lambda(t_{enter})(a/a_{eq}) & : \lambda_{FS} < \lambda < \lambda_{eq} \\ \delta_\lambda(t_{enter})\left(\frac{\lambda_{eq}}{\lambda}\right)^{2}\left(\frac{a}{a_{eq}}\right) & : \lambda > \lambda_{eq} \end{cases}$$

Thus, if we know spectrum of density-perturbations at $t_{enter}(\lambda)$, problem is fixed.

Assumption.

$\delta_\lambda(t_{enter})$ stands for function $\delta(\lambda, t_e)$ evaluated at $t = t_{enter}(\lambda)$. Therefore

$$\delta_\lambda(t_{enter}) = \delta(\lambda, t_{enter}(\lambda)) = F(\lambda)$$

Spectrum at $t_{enter}(\lambda)$

Assume power-law function:

$$\delta_\lambda(t_{enter}) = A \cdot \lambda^\alpha \propto k^{-\alpha} \propto M^{\alpha/3}$$

as a result: (not at fixed time)

$$\delta_\lambda(t) = \delta_m(t) \propto \begin{cases} 0 & : \lambda < \lambda_{FS} : M < M_{FS} \\ \lambda^\alpha \left(\frac{a}{a_{eq}}\right) \propto M^{\alpha/3}\left(\frac{a}{a_{eq}}\right) & : M_{FS} < M < M_{eq} \\ \lambda^{\alpha-2}\left(\frac{a}{a_{eq}}\right) \propto M^{\frac{\alpha}{3}-2/3}\left(\frac{a}{a_{eq}}\right) & : M > M_{eq} \end{cases}$$

Fourier Decomposition of Modes

The density contrast at (x,t) written as a superposition of modes with different wavelengths:

$$\delta(x,t) = \int \frac{d^3k}{(2\pi)^3}\, \delta_k(t)\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}$$

Strength of Perturbation : Measured by $|\delta(x,t)|^2$

$$|\delta(x,t)|^2 = \frac{1}{(2\pi)^6}\int d^3k \int d^3k' \, \delta_k(t)\, \delta_{k'}(t)\, e^{i(k-k')\cdot \underset{\sim}{x}}$$

$$\propto \int d^3k \, |\delta_k(t)|^2$$

Thus, modes in range $(k, k+d^3k)$ contribute $d^3k|\delta_k(t)|^2$ to $|\delta(x,t)|^2$

Write $\rightarrow$ $d^3k|\delta_k(t)|^2 = 4\pi k^2 dk \, |\delta_k(t)|^2 \propto d\ln k \left[k^3|\delta_k(t)|^2\right]$

thus, each logarithmic interval contributes $k^3|\delta_k|^2$ to $|\delta(x,t)|^2$: important quantity

$$k^3|\delta_k(t)|^2 \propto M^{-1}|\delta_M(t)|^2 \propto \begin{cases} 0 \; : & (M < M_{FS}) \\ M^{\frac{2\alpha}{3}-1}\left(\frac{a}{a_{eq}}\right)^2 : & (M_{FS} < M < M_{eq}) \\ M^{\frac{2\alpha}{3}-\frac{4}{3}-\frac{3}{3}}\left(\frac{a}{a_{eq}}\right)^2 : & (M > M_{eq}) \end{cases}$$

at t_{enter}

Since $\delta_k \propto k^{-\alpha} \Rightarrow |\delta_k(t_{enter})|^2 k^3 \propto k^{3-2\alpha} = \left(\frac{1}{k}\right)^{2\alpha-3}$

or $k^3|\delta_k(t_{enter})|^2 \propto M^{\frac{2\alpha}{3}-1}$ (since $\frac{1}{k} \propto M^{1/3}$)

Thus Index $\alpha = 3/2$ has special significance!

(A) $\alpha > 3/2$: $k^3|\delta_k|^2$ has more amplitude at large M at $t = t_{enter}$

(B) $\alpha < 3/2$: Most power concentrated at small Mass.

(c) when $\alpha = 3/2$: neither large nor small masses dominate. Such a power-spectrum is called a scale-invariant spectrum and is predicted by inflation & some other models of the early Universe.

If $\alpha = 3/2$ at fixed time

$$k^3 |\delta_k(t)|^2 \propto \begin{cases} 0 : & M < M_{FS} \\ \text{const} : & M_{FS} < M < M_{eq} \\ M^{-4/3} : & M > M_{eq} \end{cases}$$

Therefore shape of power spectrum depends on the ratio

$$\frac{M_{FS}}{M_{eq}} = 0.05 \, (\Omega_m h^2)^4 \left(\frac{100 \text{ eV}}{m} \right)^4$$

(1) Hot Dark Matter (neutrinos) (HDM)

$M_\nu \simeq 10\text{–}30 \text{ eV} \Rightarrow M_{FS} \sim M_{eq}$.

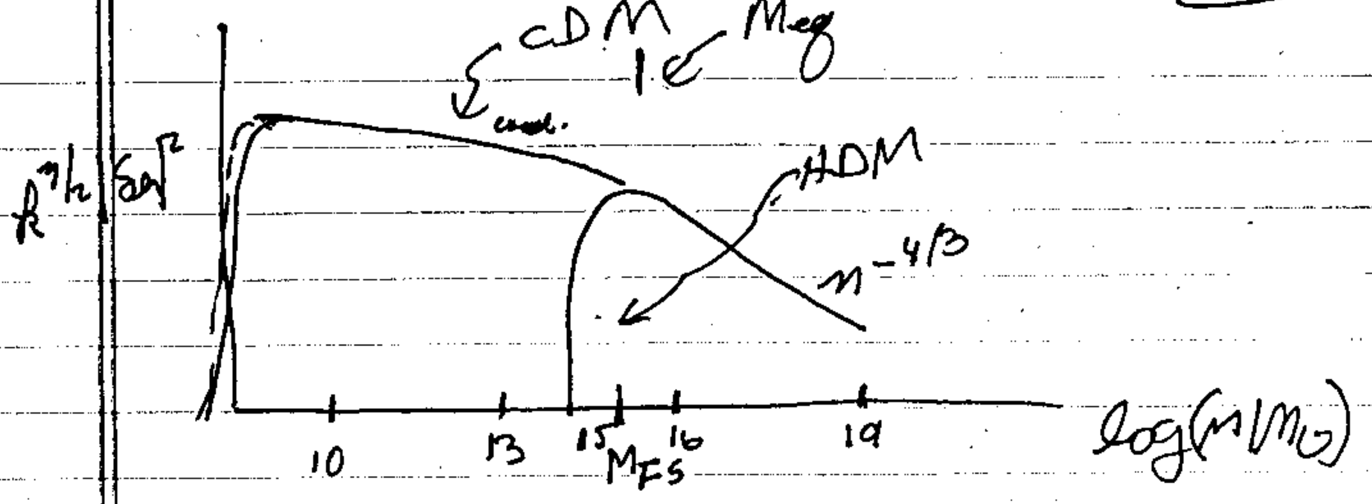
Here we predict a sharp cutoff and a peak at $M = M_{FS}$

(2) Cold Dark Matter (Wimps, etc.) (CDM).

$M_x > 1$ MeV. These cold relics lead to $M_{FS} \ll M_{eq}$.

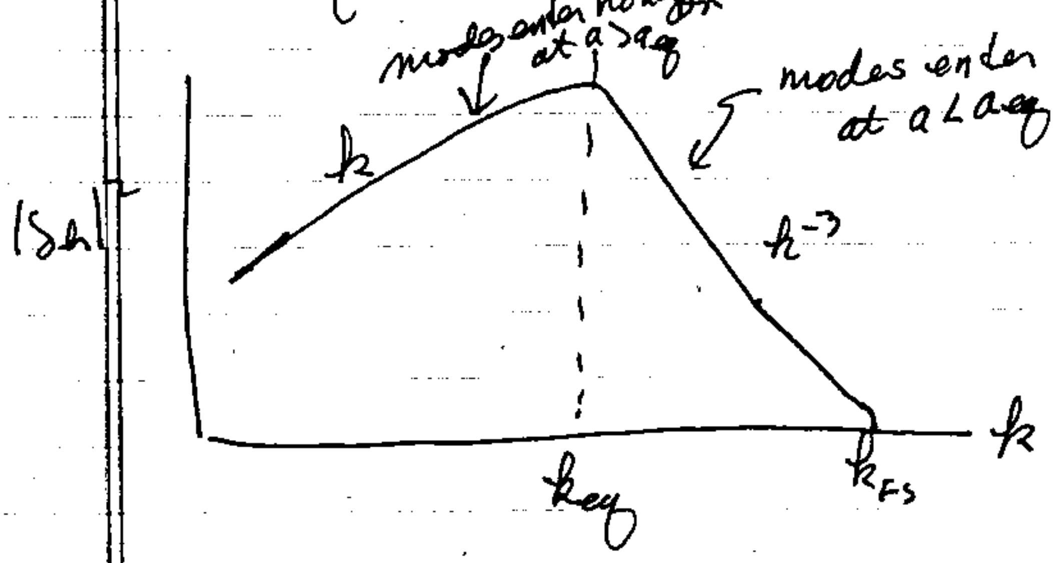
In that case spectrum is relatively flat between M_{FS} and M_{eq}. But I mentioned that δ_m can grow logarithmically in interval $t_{enter} < t < t_{eq}$ by a factor

$$G_r(\lambda) \simeq \frac{\delta(t_{eq})}{\delta(t_{enter})} = 5 \ln\left(\frac{\lambda_{eq}}{\lambda}\right) = \frac{5}{3} \ln\left(\frac{M_{eq}}{M}\right)$$



Sometimes one plots $|\delta_k|^2$ versus k

$$|\delta_k|^2 \propto \begin{cases} 0 : & k_{FS} < k \\ k^{-3}, & k_{eq} < k < k_{FS} \\ k : & k < k_{eq} \end{cases}$$



In fact $|\delta_k|^2 \propto k^n$ where $n = 4 - 2\alpha$: $k < k_{eq}$

This is shape of initial power spectrum which has not been modified by "delays" that occur when $k > k_{eq}$

Harrison-Zeldovich Spectrum

What is the power spectrum at some initial time that gave rise to $\alpha = 3/2$ Spectrum?

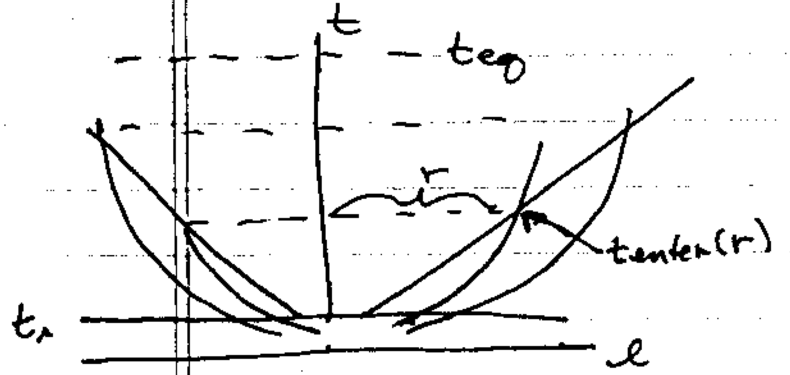
Let $\Delta^2(M) \equiv k^3 |\delta_k|^2$ or $\Delta(M) = k^{3/2} |\delta_k|$

Suppose that at some initial time t_i

$$P_k(t_i) = |\delta_k|^2 \propto k^n$$

Therefore $\Delta^2(M) \propto k^{3+n} \propto \left(\frac{1}{M}\right)^{\frac{3+n}{3}}$

$$\therefore \quad \Delta(M) \propto M^{-\left(\frac{n+3}{6}\right)}$$



Consider perturbations entering horizon $t < t_{eq}$

$$\Delta(M,t) = \Delta(M,t_i)\left(\frac{a}{a_i}\right)^2$$

or $\Delta(M,t) \propto a^2 \cdot M^{-\left(\frac{n+3}{6}\right)}$

(1) Perturbation with physical scale r come through horizon when $l_H(t) = r$ or $2ct = r$

(2) Mass contained in perturbation when it entered is just horizon mass at t_{enter}

$$M_H \approx \rho_{DM} r^3 \approx \rho_{DM} (ct)^3$$

In RD phase: $t \propto a^2$. Since DM is nr $\rho_D \propto a^{-3}$

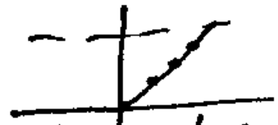
Therefore $M_H \propto a^{-3}(a^2)^3 \propto a^3$

Thus scale factor at t_{enter} is $a \propto M_H^{1/3}$

(3) at $t = t_{enter}$

$$\Delta(M_H) \propto M_H^{2/3} \cdot M_H^{-\left(\frac{n+3}{6}\right)} = M_H^{\left(\frac{4-n-3}{6}\right)}$$

$$\boxed{\Delta(M) \propto M^{-\left(\frac{n-1}{6}\right)}}$$



spectrum along light cone

Thus if initial power spectrum index $n=1$, all fluctuations would have same rms amplitude when they entered

Comments

(A) at given initial time t_i

$$\Delta(M, t_i) \propto M^{-\left(\frac{n+3}{6}\right)} \propto M^{-4/3}$$ {spectrum at $t = t_i$

$M < M_{eq}$ • Mass spectrum modified at $M < M_{eq}$ because of delay. Even though with more power low masses entered earlier, they get delayed longer and so Δ for larger masses catch up and all are $=$ at t_{eq}

$M > M_{eq}$ • No modification since they enter at $M > M_{eq}$. And initial $M^{-4/3}$ spectrum preserved!