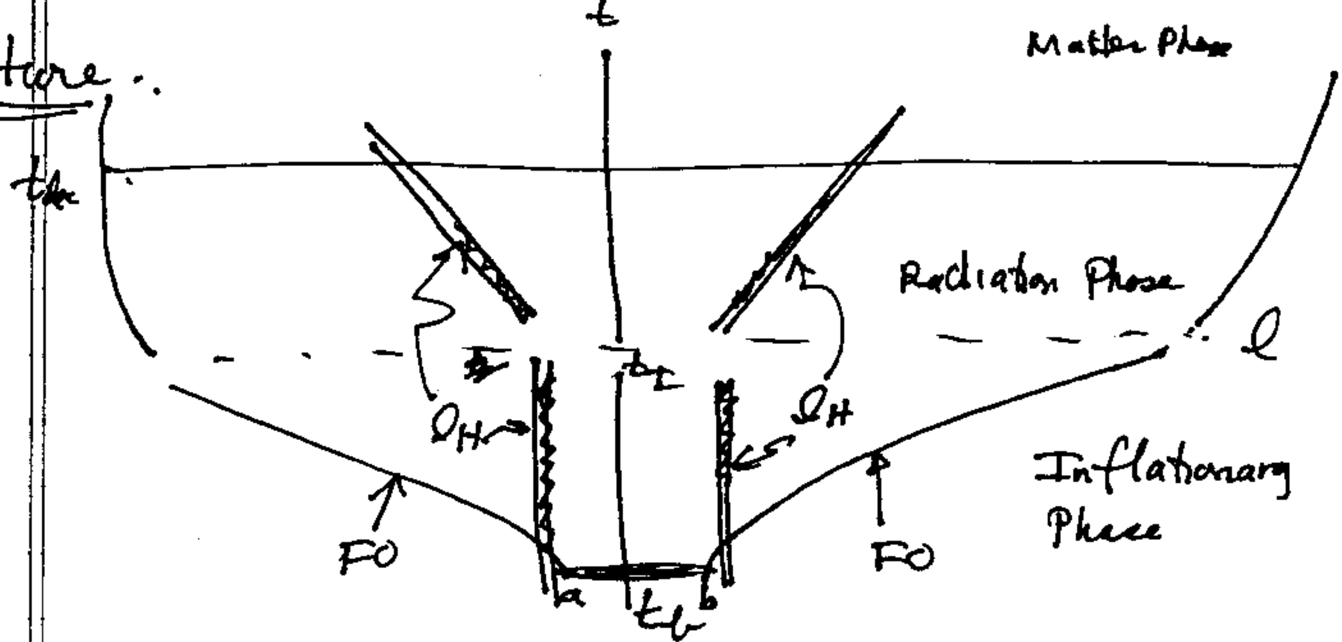
Recap — Inflation

Picture..



(1) Exponential expansion during inflationary period accelerates initial patch outside initial ~~.....~~ horizon.

(2) Explains why patches outside horizon at decoupling were initially in causal contact.

(3) This helps to explain big mystery.. why do regions with scales ~~beyond~~ large compared to horizon radius at decoupling have same T to 1 part in 10^5. This is a big big problem since ∵ thermal character of black-body radiation established only after relaxation process

- remove T-spatial gradients
- " spectral distortions.

Quantitative:

(A) Assume temperature at start of inflation, $t = t_b$ $kT \simeq 10^{14}$ GeV (spontaneous symmetry breaking scale) corresponds to $T \simeq 10^{27}$ K

$t \simeq H^{-1} \simeq 10^{-34}$ sec

(B) Next assume $H(t_I - t_b) \simeq 100$

Duration of inflation $\Delta t = t_I - t_b \simeq 10^{-32}$ s.

(C) Let initial patch of matter: $d \simeq l_H = \frac{c}{H}$

$d_b \simeq 3\times 10^{10-34} \simeq 3\times 10^{-24}$ cm.

(d) At end of inflation:

$$d_I = e^{H\Delta t}\cdot d_b \simeq e^{100}\times 3\times 10^{-24} = 3\times 10^{43} \times 3\times 10^{-24}$$

$\simeq 10^{20}$ cm

(e) what is the size of that patch today?

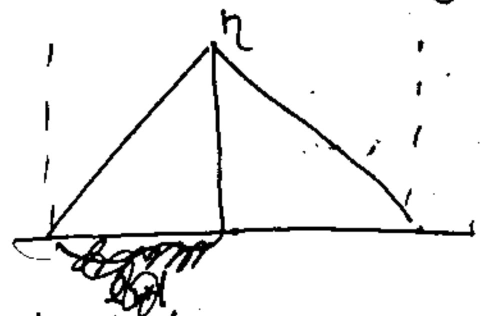
$$1 + z_I = \frac{T_I}{T_0} = \frac{10^{27} K}{2.728} = 4\times 10^{26}$$

as patch evolves like FO, today its scale would be

$$d(t_0) = d_0 = \frac{a_0}{a_I}\cdot d_I = (1+z_I)\, d_I$$

$$d(t_0) \simeq 4\times 10^{26}\times 10^{20} \simeq 4\times 10^{46} \text{ cm!}$$

(f) Current Horizon!



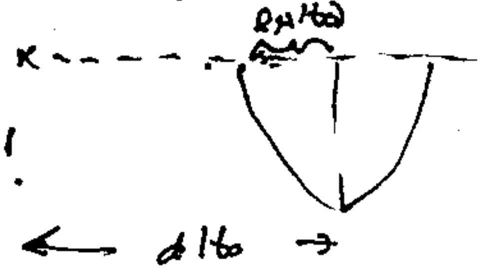
causal contact

$l_H(t_0) \simeq 3ct_0$

$t_0 \simeq 13.7\times 10^{9}$ y

$l_H(t_0) \simeq 4\times 10^{28}$

Thus all the matter we can currently observe makes up a tiny fraction of matter that was in causal contact during inflation.



$$\frac{d(t_0)}{\ell_H(t_0)} \simeq \frac{4\times10^{46}}{4\times10^{28}} \simeq 10^{18}!$$

$$\frac{V(t_0)}{\ell_H(t_0)} \simeq 10^{54}!$$

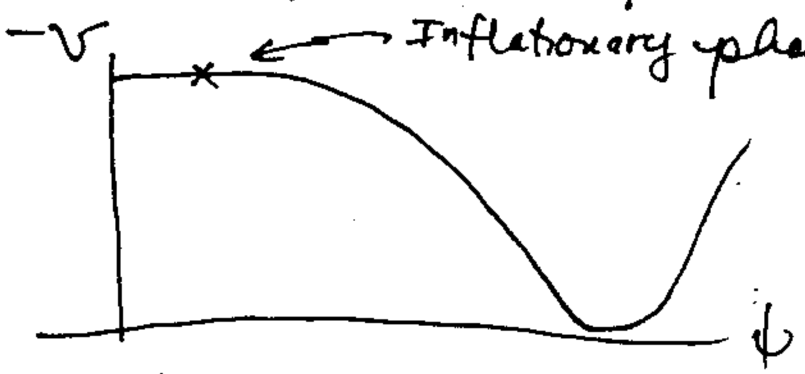
Dynamics

Inflation driven by negative pressure exerted by vacuum energy density of a scalar field, ϕ.

Recall: $$\boxed{\begin{aligned} P &\simeq \tfrac{1}{2}\dot\phi^2 - V(\phi) \\ \rho &= \tfrac{1}{2}\dot\phi^2 + V(\phi) \end{aligned}}$$

During initial "slow roll" phase $\dot\phi^2 \ll |V(\phi)|$ in which case:

$$P \simeq -V(\phi) \;.\; \rho \simeq +V(\phi) \Rightarrow P \simeq -\rho$$



Conservation: (1) $\boxed{\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P)}$ (A)

In this case: $\Rightarrow$ $\dot{\rho} = 0$, $\rho = \text{const}$

Friedmann Eq.

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

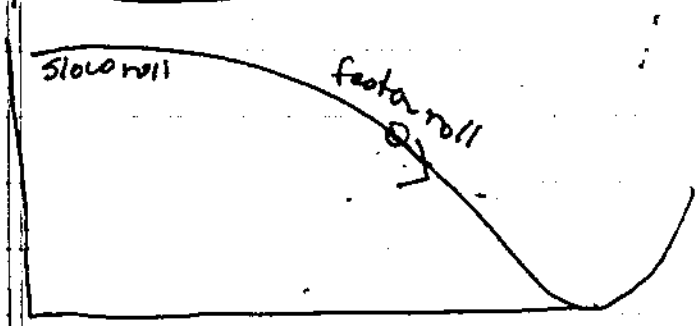
Eq. of state $\Rightarrow$ $\ddot{a} = +\frac{8\pi G}{3}\rho = \text{const}$

$\ddot{a} > 0$: $\ddot{a} = H^2$

$\Rightarrow a(t) \propto e^{Ht}$ exponential expansion.

also explains $l_H = \frac{1}{H} = \text{const}$ horizon.

End of inflation:



Eq. (A) $\Rightarrow$ $\boxed{\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi)} = 0$ (B)

eq. (1)

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + V\right) + 3H\left(\dot{\phi}^2\right) = 0$$

$$\frac{2}{2}\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi} + 3H\dot{\phi}^2 = 0$$

$$\dot{\phi}\left[\ddot{\phi} + V' + 3H\dot{\phi}\right] = 0$$

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V' = 0} \quad (C)$$

ϕ damping analogous to ball rolling down a hill with friction.

__Coupling__: "Inflanton" ~~field~~ $\underbrace{}^{\text{field } \phi}$ couples to fields of matter and radiation: energy density of inflaton field or ϕ quanta decay at rate Γ into particles. At this point (reheating) entropy observed in present Universe was generated.

~~Energy density ρ_M of particles into which ϕ decayed satisfies a conservation law like eq. (B), but corrected to account for dissipation of ϕ~~

~~Matter~~ ~~$\boxed{\dot{\rho}_M + 3H(\rho_M + P_M) = \underbrace{\Gamma\rho_\phi}_{\text{coupling}}}$ (C)~~

Field : when field starts to oscillate (damped) around minimum of potential one finds that

$$\boxed{\rho_\phi(t) = \rho_\phi(t_I) \cdot \left(\frac{a(t_I)}{a(t)}\right)^3 e^{-\Gamma(t-t_I)}} \quad (D)$$

where Γ is rate of decay of ϕ quanta into particles, and t_I is beginning time of inflation oscillation and decay.

Matter : Energy density ρ_M of particles into which ϕ decayed satisfies a conservation law like eq. (B), but corrected to account for dissipation

$$\dot{\rho}_M + 3H(t)(\rho_M + P_M) = \underbrace{\Gamma \rho_\phi}_{\text{production term}}$$

Assume particles produced are relativistic
$P_M = \rho_M / 3$

In that case

$$\dot{\rho}_M + 3H(t)\left(\tfrac{4}{3}\rho_M\right) = \Gamma \rho_\phi$$

$$\dot{\rho}_M + 4\frac{\dot{a}}{a}\rho_M = \Gamma \rho_\phi$$

Rewrite: $\boxed{\dot{\rho}_M + \frac{4\dot{a}}{a}\rho_M = \Gamma \rho_\phi}$ (E)

Solve with integrating factor:

$$\frac{d}{dt}\left(e^{4\ln a}\rho_M\right) = e^{4\ln a}\,\Gamma\rho_\phi$$

Integrate from t_I (where $\rho_M = 0$) $\to t$

$$e^{4\ln a}\rho_M = \int_{t_I}^{t} dt'\, e^{4\ln a}\,\Gamma\rho_\phi$$

from (D)

$$a^4(t)\rho_M(t) = \int_{t_I}^{t} dt'\, a^4(t')\,\Gamma \left\{ \rho_\phi(t_I)\left(\frac{a(t_I)}{a(t')}\right)^3 e^{-\Gamma(t'-t_I)} \right\}$$

$$\Rightarrow \boxed{\rho_M(t) = \frac{\rho_\phi(t_I)\,\Gamma\, a^3(t_I)}{a^4(t)}\int_{t_I}^{t'} dt'\, a(t')\, e^{-\Gamma(t'-t_I)}}$$

~~Limit $\Gamma \gg (t - t_I)$~~

~~$\rho_M(t) = \rho_\phi(t_I)$~~

$$\text{Lit } Q = \int_{t_I}^{t} dt' a(t') e^{-\Gamma(t'-t_I)} = e^{+\Gamma(t_I)} \underbrace{\int_{t_I}^{t} dt\, a(t) e^{-\Gamma t'}}_{Q}$$

Integrate by parts: $u = a(t')$; $dv = dt\, e^{-\Gamma t'}$

$du = da$, $V = -\frac{1}{\Gamma} e^{-\Gamma t'}$

$$\therefore Q = \left[a(t') \left(-\frac{1}{\Gamma} e^{-\Gamma t'} \right) \right]_{t_I}^{t} + \int_{a_I}^{a(t)} \frac{1}{\Gamma} e^{-\Gamma t'} da$$

$$= \frac{1}{\Gamma} \left\{ -a(t) e^{-\Gamma t} + a(t_I) e^{-\Gamma t_I} + \underbrace{\int_{t_I}^{t} e^{-\Gamma t'} \frac{da}{dt'} dt'}_{Q_{(1)}} \right\}$$

$$Q_{(1)} = \int_{t_I}^{t} e^{-\Gamma t'} H(t') a(t') dt'$$

$dv = e^{-\Gamma t'} dt'$; $u = H(t') a(t')$

$V = -\frac{1}{\Gamma} e^{-\Gamma t'}$: $d(Ha)$

$$Q_{(1)} = \left[H(t') a(t') \left(-\frac{1}{\Gamma} e^{-\Gamma t'} \right) \right]_{t_I}^{t} + \int_{t_I}^{t} \frac{1}{\Gamma} e^{-\Gamma t'} d(Ha)$$

$$= \frac{1}{\Gamma} \left[-H(t) a(t) e^{-\Gamma t} + H(t_1) a(t_1) e^{-\Gamma(t_1)} \right] + Q_{(2)}$$

$$\therefore Q = \frac{1}{\Gamma} \left\{ -a(t) e^{-\Gamma t} + a(t_I) e^{-\Gamma t_I} + \frac{1}{\Gamma} \left[-H(t) a(t) e^{-\Gamma t} + H(t_1) a(t_1) e^{-\Gamma t_1} \right] + Q_{(2)} \right.$$

$$\rho_m(t) = \frac{\rho_\phi(t_1)\, a^3(t_I)\, \Gamma}{a^4(t)} \times \frac{\cancel{e}}{\Gamma} \times \Big[-a(t)\underbrace{e^{-\Gamma(t-t_I)}}_{0} + a(t_I)$$

$t - t_I = \frac{s}{H} \Rightarrow \Gamma(t-t_1) = \frac{\Gamma}{H} s \gg 1$

$$+ \frac{1}{\Gamma}\Big[-H(t)a(t)\underbrace{e^{-\Gamma(t-t_I)}}_{0} + H(t_1)a(t_1) \Big] + \cdots$$

$$\rho_m(t) = \frac{\rho_\phi(t_1)\, a^3(t_I)}{a^4(t)} \left\{ a(t_1) + \frac{a(t_1)H(t_1)}{\Gamma} + \cdots \right\}$$

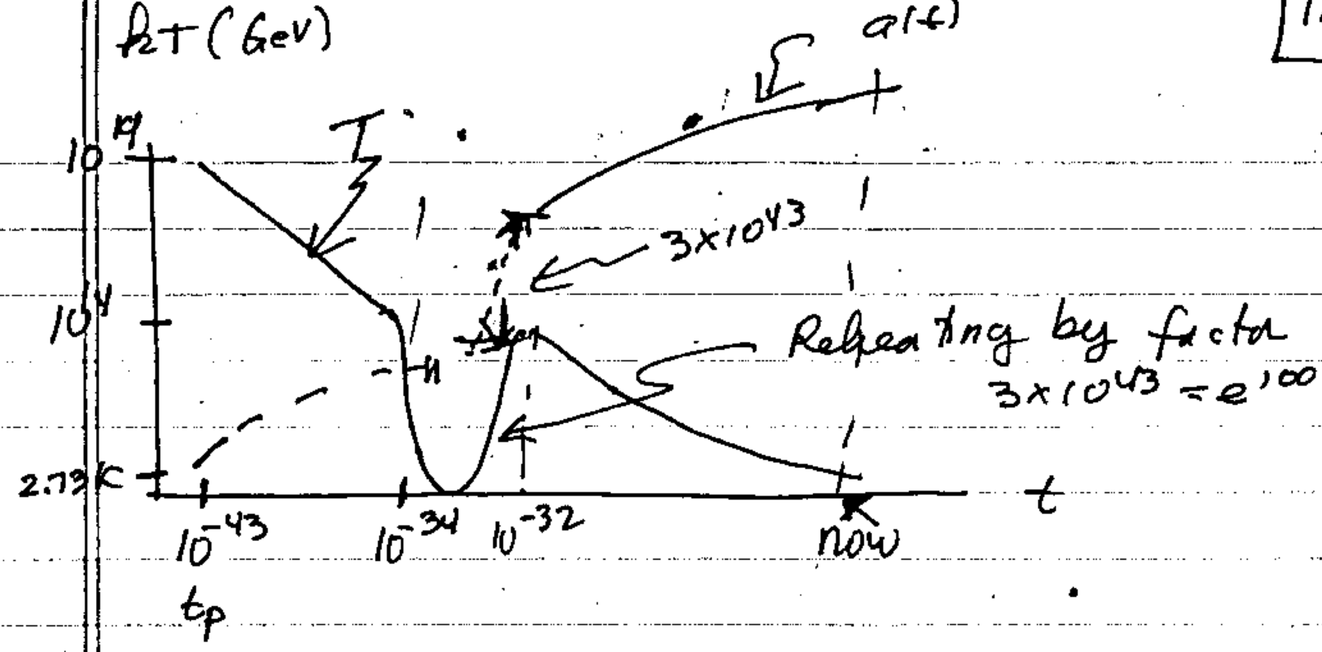
$$\boxed{\rho_m(t) = \rho_\phi(t_1) \left(\frac{a(t_1)}{a(t)} \right)^4 \left[1 + \frac{H(t_1)}{\Gamma} + \cdots \right]}$$

Interesting result:

ρ_m jumps immediately to value $\rho_\phi(t_I)$ and then decreases in usual a^{-4} factor for relativistic particles.

All the energy of inflaton field at the end of inflation went into ordinary matter & radiation.

Visual Summary ⟶

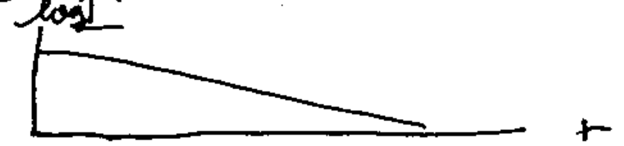


Cold Dark Matter

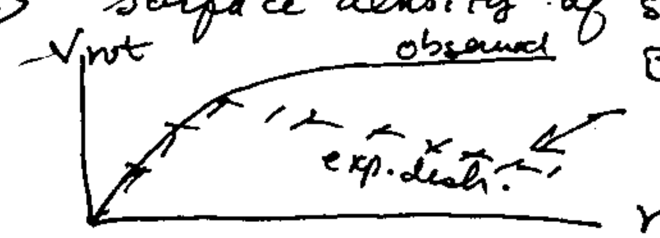
Since $\Omega_m \simeq 0.3$ and $\Omega_B h^2 = 0.02$, we conclude that most of the mass in the Universe is not in the form of ordinary baryons. If particles making up most of the mass in the Universe are not baryons, what are they?

- Dark: we dont see it:

Rotation curves of galaxies ~~xxx~~ example:
Spiral disk visual light falls off exponentially with radius



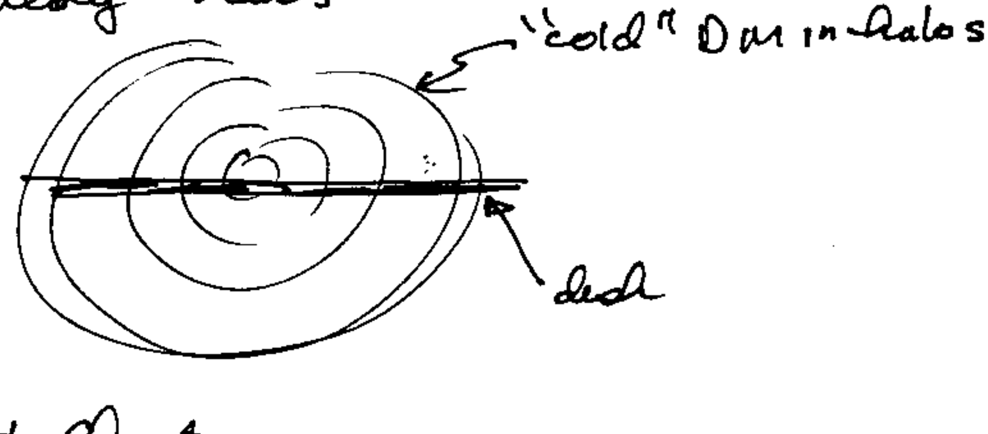
$\Rightarrow$ Surface density of stars: $\Sigma \propto \exp(-\alpha r)$



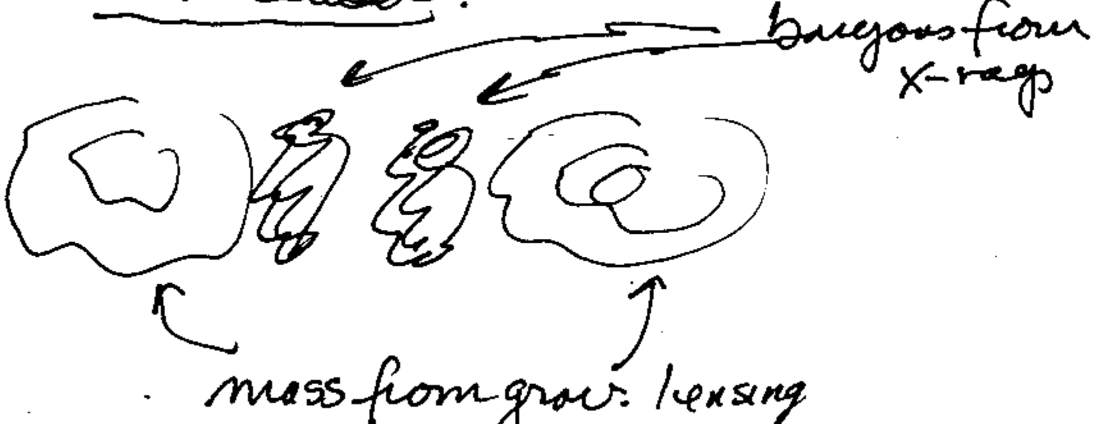
But implied mass cannot account for V_{ROT}

• Dark matter not dissipative to fall into disks of galaxies. More likely to be non-relativistic (cold) particles occupying galaxy halos

$$\frac{v^2}{r} = \frac{GM(r)}{r^2}$$



• Bullet Cluster:



Relics (WIMPS: weakly interacting massive particles).

Idea: at sufficiently high density (and high redshift) these particles are in thermal contact with rest of thermal plasma. They remain in equilibrium via pair annihilation ~~as~~ with their antiparticles and inverse pair creation.

Their abundance freezes out when $kT < m_x c^2$ (m_x is WIMP mass)

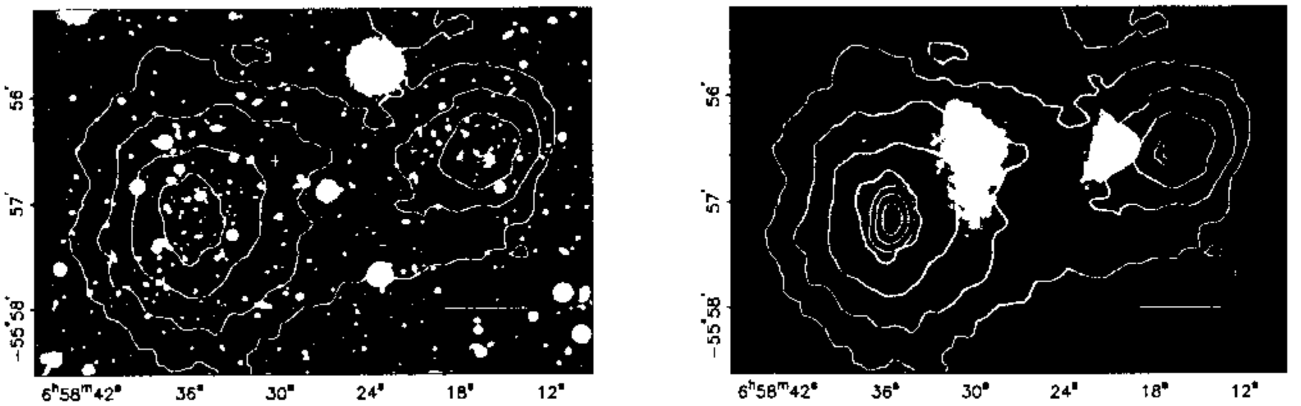


FIG. 1.—*Left panel*: Color image from the Magellan images of the merging cluster 1E 0657−558, with the white bar indicating 200 kpc at the distance of the cluster. *Right panel*: 500 ks *Chandra* image of the cluster. Shown in green contours in both panels are the weak-lensing κ reconstructions, with the outer contour levels at $\kappa = 0.16$ and increasing in steps of 0.07. The white contours show the errors on the positions of the κ peaks and correspond to 68.3%, 95.5%, and 99.7% confidence levels. The blue plus signs show the locations of the centers used to measure the masses of the plasma clouds in Table 2.

Both peaks are offset from their respective BCGs by $\sim 2\,\sigma$ but are within 1 σ of the luminosity centroid of the respective component's galaxies (both BCGs are slightly offset from the center of galaxy concentrations). Both peaks are also offset at $\sim 8\,\sigma$ from the center of mass of their respective plasma clouds. They are skewed toward the plasma clouds, and this is expected because the plasma contributes about one-tenth of the total cluster mass (Allen et al. 2002; Vikhlinin et al. 2006) and a higher fraction in nonstandard gravity models without dark matter. The skew in each κ peak toward the X-ray plasma is significant even after correcting for the overlapping wings of the other peak, and the degree of skewness is consistent with the X-ray plasma contributing $14\%^{+9\%}_{-8\%}$ of the observed κ in the main cluster and $10\%^{+12\%}_{-10\%}$ in the subcluster (see Table 2). Because of the large size of the reconstruction (34′ or 9 Mpc on a side), the change in κ due to the mass-sheet degeneracy should be less than 1%, and any systematic effects on the centroid and skewness of the peaks are much smaller than the measured error bars.

The projected cluster galaxy stellar mass and plasma mass within 100 kpc apertures centered on the BCGs and X-ray plasma peaks are shown in Table 2. This aperture size was chosen because smaller apertures had significantly higher κ measurement errors and because larger apertures resulted in a significant overlap of the apertures. Plasma masses were computed from a multicomponent three-dimensional cluster model fit to the *Chandra* X-ray image (details of this fit will be given elsewhere). The emission in the *Chandra* energy band (mostly optically thin thermal bremsstrahlung) is proportional to the square of the plasma density, with a small correction for the plasma temperature (also measured from the X-ray spectra), which gives the plasma mass. Because of the simplicity of this cluster's geometry, especially at the location of the subcluster, this mass estimate is quite robust (to a 10% accuracy).

Stellar masses are calculated from the I-band luminosity of all galaxies equal in brightness or fainter than the component BCG. The luminosities were converted into mass by assuming (Kauffmann et al. 2003) $M/L_I = 2$. The assumed mass-to-light ratio is highly uncertain (and can vary between 0.5 and 3) and depends on the history of the recent star formation of the galaxies in the apertures; however, even in the case of an extreme deviation, the X-ray plasma is still the dominant baryonic component in all of the apertures. The quoted errors are only the errors on measuring the luminosity and do not include the uncertainty in the assumed mass-to-light ratio. Because we did not apply a color selection to the galaxies, these measurements are an upper limit on the stellar mass since they include contributions from galaxies not affiliated with the cluster.

The mean κ at each BCG was calculated by fitting a two-peak model, each peak circularly symmetric, to the reconstruction and subtracting the contribution of the other peak at that distance. The mean κ for each plasma cloud is the excess κ after subtracting off the values for both peaks.

The total of the two visible mass components of the subcluster is greater by a factor of 2 at the plasma peak than at the BCG; however, the center of the lensing mass is located near the BCG. The difference in the baryonic mass between these two positions would be even greater if we excluded the contribution of the nonpeaked plasma component between the

TABLE 2
COMPONENT MASSES

Component	R.A. (J2000)	Decl. (J2000)	M_X ($10^{12}\,M_\odot$)	M_* ($10^{12}\,M_\odot$)	$\bar{\kappa}$
Main cluster BCG	06 58 35.3	−55 56 56.3	5.5 ± 0.6	0.54 ± 0.08	0.36 ± 0.06
Main cluster plasma	06 58 30.2	−55 56 35.9	6.6 ± 0.7	0.23 ± 0.02	0.05 ± 0.06
Subcluster BCG	06 58 16.0	−55 56 35.1	2.7 ± 0.3	0.58 ± 0.09	0.20 ± 0.05
Subcluster plasma	06 58 21.2	−55 56 30.0	5.8 ± 0.6	0.12 ± 0.01	0.02 ± 0.06

NOTES.—Units of right ascension are hours, minutes, and seconds, and units of declination are degrees, arcminutes, and arcseconds. All values are calculated by averaging over an aperture of 100 kpc radius around the given position (marked with blue plus signs for the centers of the plasma clouds in Fig. 1); $\bar{\kappa}$ measurements for the plasma clouds are the residuals left over after the subtraction of the circularly symmetric profiles centered on the BCGs.

WIMP Scenario:

$$X + \bar{X} \rightleftarrows 2\gamma$$

← massless, not necessarily photons.

Annihilation rate per particle ~~and~~ of particles and anti-particles:

$$n_x \langle \sigma v \rangle$$

n_x = number density
$\langle \sigma v \rangle$ = product of annihilation cross-section and ~~rel~~ relative velocity.

Rate of decrease of such particles in comoving volume a^3 is then

$$n_x a^3 * n_x \langle \sigma v \rangle$$

Creation rate (n-independent) of $X, \bar{X}$ pairs from thermal background of γ (light particles). But this must balance annihilation rate in equilibrium when $kT \gg m_x c^2$

Creation rate = annihilation rate when $n_x = n_{eq}$ is density of these particles in equilibrium.

$\therefore$

Creation rate $= n_{eq}^2 a^3 \langle \sigma v \rangle$

$\therefore$ Net change of particles in a^3 volume

$$\frac{d}{dt}(n_x a^3) = \text{Creation rate} - \text{Annihilation rate}$$
$$= n_{eq}^2 a^3 \langle \sigma v \rangle - n_x^2 a^3 \langle \sigma v \rangle$$

Therefore

$$\frac{d}{dt}(n_x a^3) = \langle\sigma v\rangle a^3 \left[n_{eq}^2 - n_x^2\right]$$

or $$\boxed{\frac{1}{a^3}\frac{d}{dt}(n_x a^3) = \langle\sigma v\rangle \left[n_{eq}^2 - n_x^2\right]} \quad (F)$$

Recall: $T \propto a^{-1} \Rightarrow$

LHS: $$\frac{1}{a^3}\frac{d}{dt}(n_x a^3) = \frac{1}{a^3}\frac{d}{dt}\left[T^3\left(\frac{n_x a^3}{T^3}\right)\right] = \frac{(Ta)^3}{a^3}\frac{d}{dt}\left(\frac{n_x}{T^3}\right)$$

Eq. (F) becomes

$$T^3 \frac{d}{dt}\left(\frac{n_x}{T^3}\right) = \langle\sigma v\rangle \left[n_{eq}^2 - n_x^2\right]$$

RHS: mult $\frac{T^6}{T^6}$

$$\frac{d}{dt}\left(\frac{n_x}{T^3}\right) = \frac{\langle\sigma v\rangle}{T^3} \times T^6 \left[\left(\frac{n_{eq}}{T^3}\right)^2 - \left(\frac{n_x}{T^3}\right)^2\right]$$

Let $Y \equiv \frac{n_x}{T^3} \Rightarrow$

$$\boxed{\frac{dY}{dt} = T^3 \langle\sigma v\rangle \left\{Y_{eq}^2 - Y^2\right\}} \quad (G)$$

where $$Y_{eq} = \frac{n_{eq}}{T^3}$$

Let $x = \frac{m_x c^2}{k_B T}$

(1) at high temperatures; $k_B T \gg m_x c^2$
$x \ll 1$. Detailed balance between pair creation and annihilation leads to $Y \to Y_{eq}$.

(2) Nature of equilibrium distribution:

$n_{eq} = \int n(p) dp$ where

$$n(p) = \frac{g_i 4\pi p^2 dp}{h^3} \times \frac{1}{\exp\left(\frac{E}{k_B T}\right) \pm 1}$$

where $E = \sqrt{(mc^2)^2 + p^2 c^2}$

Limits: (1) $k_B T \gg mc^2$

$$n_{eq} \simeq g_i \left(\frac{k_B T}{c \hbar}\right)^3$$

(2) $k_B T \ll mc^2$

$$n_{eq} = g_i \left(\frac{m k_B T}{2\pi \hbar^2}\right)^{3/2} e^{-\frac{mc^2}{k_B T}}$$

$x \gg 1$ $\therefore$ when $k_B T \ll mc^2$, Y_{eq} becomes exponentially suppressed.

Let $T_x \equiv \frac{m_x c^2}{k_B} \Rightarrow x = \frac{T_x}{T}$

or since $T(a) a = \text{const} \Rightarrow \frac{T_x}{T} = \frac{a}{a_x}$: $a_x = a(T_x)$

therefore: $\frac{dx}{dt} = \frac{\dot{a}}{a_x} = \frac{a\,H(a)}{a_x}$

Since $\frac{a}{a_x} = x$ we have $\frac{dx}{dt} = x \cdot H$

During radiation era: $a \propto t^{1/2} = \text{const} \cdot t^{1/2}$

$$H = \frac{\dot{a}}{a} = \frac{1}{2t}$$

But $t \propto a^2 \propto \frac{1}{T^2}$

$$\Rightarrow \frac{t}{t_x} = \left(\frac{T_x}{T}\right)^2 \quad \text{or} \quad \begin{cases} t_x: \text{time when} \\ T = T_x = \frac{m_x c^2}{k_B} \end{cases}$$

$$H = \frac{1}{2 t_x}\left(\frac{T}{T_x}\right)^2$$

$$\boxed{\therefore H(x) = H(m_x)/x^2} \quad : H(m_x) = H(t = t_x)$$

From eq. (G): $\frac{dY}{dt} = \frac{dx}{dt}\frac{dY}{dx}$

LHS $\therefore$ $\frac{dY}{dt} = H(x) \times \frac{dY}{dx} = \left[\frac{H(m_x)}{x^2} \cdot x\right]\frac{dY}{dx} = \frac{H(m)}{x}\frac{dY}{dX}$

RHS : $T = \frac{m_x c^2}{k_B \cdot x} = \frac{T_x}{x}$

$$\frac{H(m)}{x}\frac{dY}{dX} = \frac{T_x^3}{x^3}\langle\sigma v\rangle\left(Y_{eq}^2 - Y^2\right)$$

$$\boxed{\frac{dY}{dx} = -\frac{\lambda}{x^2}\left(Y^2 - Y_{eq}^2\right)} \quad (H)$$

$$\lambda = \frac{T_x^3 \langle \sigma v \rangle}{H(m_x)} \quad \left\{ \begin{array}{l} \propto \text{Ratio of annihilation} \\ \text{rate to expansion rate} \end{array} \right\}$$

(i) at Late times $Y_{eq} \propto e^{-\frac{T_x}{T}}$

Equilibrium abundance drops so low that at freeze-out $Y \gg Y_{eq}$.

$$\frac{dY}{dx} \simeq -\frac{\lambda Y^2}{x^2} \qquad x \gg 1$$

Integrate from $x = x_F$ (Freezeout) to $x = \infty$

$$\int_{Y_F}^{Y_\infty} \frac{dY}{Y^2} = -\lambda \int_{x_F}^{\infty} \frac{dx}{x^2}$$

$$\left[-\frac{1}{Y}\right]_{Y_F}^{Y_\infty} = -\lambda \left[-\frac{1}{x}\right]_{x_F}^{\infty}$$

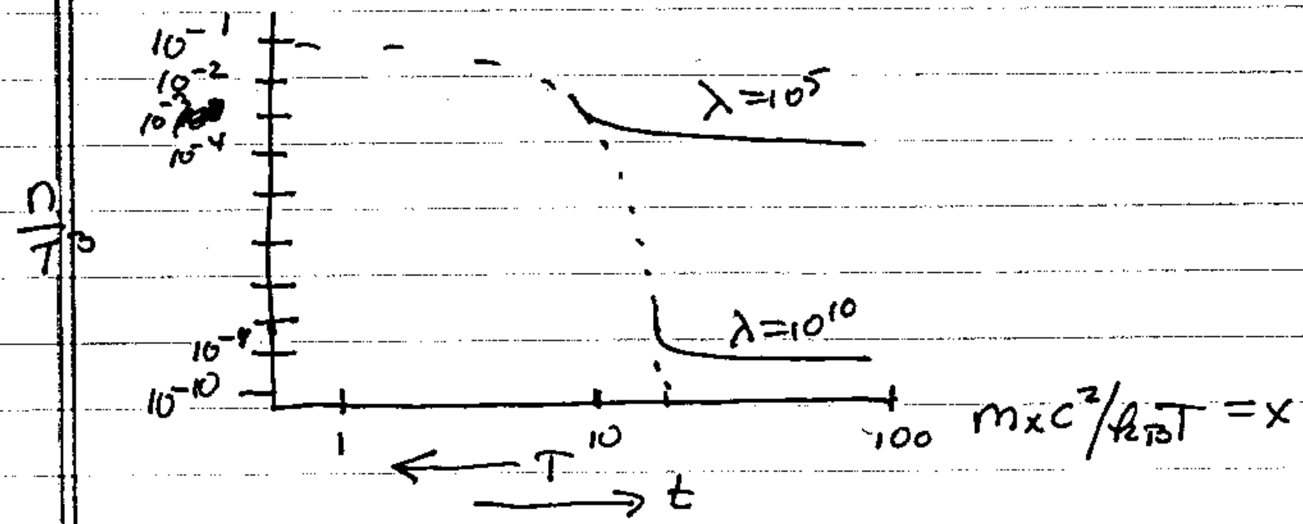
$$-\frac{1}{Y_\infty} + \frac{1}{Y_F} = -\lambda \left[\frac{1}{x_F}\right]$$

$$\boxed{\frac{1}{Y_\infty} - \frac{1}{Y_F} = \frac{\lambda}{x_F}} \quad (I)$$

Typically $Y_F \gg \frac{1}{Y_\infty}$, so $\boxed{Y_\infty \simeq \frac{x_f}{\lambda}}$

Typicall $x_f \simeq 10$; i.e., freeze out occurs when T drops to $\sim \left(\frac{1}{10}\right) \times \frac{m_x c^2}{k_B}$ (like Saha)

Real solutions (numerical)



- Abundances track equilibrium curves until $x \simeq x_f = 10$ at which freeze-out occurs

- $Y_\infty \simeq \frac{x_f}{\lambda} \simeq \frac{10}{\lambda}$ is about correct

- Particles with larger σ (λ) have later freeze out, lower abundance. Tend to be higher masses - higher cross section

Current Abundance:

at time of Y_∞: $a = a_1$: defines epoch t_1
$n_\bullet(a_1) = Y_\infty T_1^3$

: Current mass-density $\cancel{\rho = m_x n(a_0) = m_x Y_\infty \left(\frac{a_1}{a_0}\right)^3}$

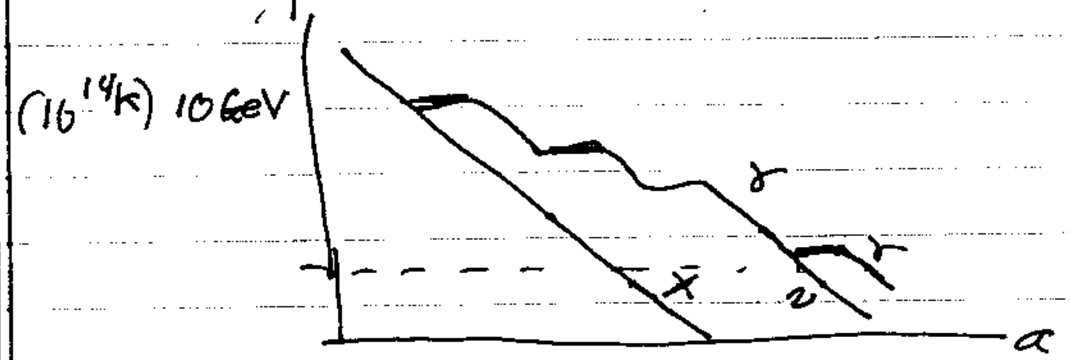
Current number density: $n_x(0) = \left(\frac{a_1}{a_0}\right)^3 n_x(a_1)$

Current mass density: $\rho_x(0) = m_x n_x(0)$

$$\therefore\ \rho_x(0) = m_x \left(\frac{a_1}{a_0}\right)^3 Y_{\infty} T_1^3 \qquad \left(T_1 = \frac{T_1}{T_0} \times T_0\right)$$

$$\text{or}\quad \rho_x(0) = m_x Y_{\infty} \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 T_0^3$$

Note: $a_1 T_1 \neq a_0 T_0$ because photons are heated by annihilation of 200 particles with $mc^2 = \frac{100}{T}$ GeV $\to$ 1 MeV (like ν adiabat.)



Turns out that $(a_1 T_1 / a_0 T_0)^3 \simeq \frac{1}{30}$

$$\Rightarrow \rho_x(0) = \frac{m_x Y_{\infty} T_0^3}{30} \qquad Y_{\infty} = \frac{x_f}{\lambda}$$

$$\Omega_x = \frac{\rho_x(0)}{\rho_{crit}} = \frac{x_f}{\lambda} \cdot \frac{m_x T_0^3}{30 \cdot \rho_{crit}}$$

Since $\lambda = \frac{m_x^3 \langle \sigma v \rangle}{H(m)}$

$$\Omega_x = \frac{x_f \cdot H(m)}{m_x^3 \langle \sigma v \rangle} \frac{m_x T_0^3}{30 \rho_{crit}} = \frac{H(m)\ x_f T_0^3}{30\ m_x^2 \langle \sigma v \rangle \rho_{crit}}$$

Find $H(m_x)$; i.e., Hubble rate when $T = T_x = \frac{m_x c^2}{k_B}$

Recall: $H^2_{(m_x)} = \frac{8\pi G}{3} \rho(T_x)$

Energy density: $u = \rho(T_x) c^2 = g_s a_B T_x^4$

$$\therefore \rho(T_x) = g_s \cdot \left(\frac{8\pi^5 k_B^4}{15 h^3 c^3}\right) \times \left(\frac{m_x c^2}{k_B}\right)^4 \Big/ c^2$$

$$\therefore \rho(T_x) = \frac{g_s \cdot 8\pi^5 m_x^4 c^3}{15 h^3}$$

$$\therefore H^2 = g_s \cdot \frac{64\pi^6 G}{45 h^3} m_x^4 c^3 \quad \text{or} \quad h = 2\pi\hbar$$

$$H^2 = \frac{g_s \cdot 8\pi^3 G c^3 m_x^4}{45 \hbar^3}$$

$$\Rightarrow H = \left[\frac{8\pi^3 \cdot g_s(m_x) c^3}{45 \hbar^3}\right]^{1/2} m_x^2$$

From previous page we have

$$\boxed{\Omega_x = \left[\frac{8\pi^3 \cdot g_s(m_x) c^3}{45 \hbar^3}\right]^{1/2} \cdot \frac{x_F T_0^3}{30 \langle \sigma v \rangle \rho_{crit}}}$$

Can $\Omega_x \simeq \Omega_m \simeq 0.3$?

at $T \simeq 100$ GeV

$g_s(m)$ equals contributions from all particles

Put numbers:

(1) Standard Model: at $T_x \simeq 10^2$ GeV $g_s(m_x)$ includes contributions from all particles in standard model (3 generations of quarks & leptons, photons, gluons, weak bosons, ...)

$$g_s \simeq 100$$

$$\Omega_x \simeq \frac{0.3}{h^2}\left(\frac{x_f}{10}\right)\left(\frac{g(m_x)}{100}\right)^{1/2}\left(\frac{10^{-39}}{\langle\sigma v\rangle}\right)$$

Fact that $\Omega_x \sim 1$ for cross-sections $\sim 10^{-39}\,\mathrm{cm^3\,s^{-1}}$ is encouraging since $\exists$ several theories which predict WIMP cross-sections this small.

(2) Cold-Dark Matter

Summarize: WIMPS and other candidates that become non-relativistic at high z are known as cold-dark-matter.

(3) Hot & Warm Dark Matter

Neutrinos and other like light particles only go non-relativistic at recent epochs owing to low masses. Since they stay relativistic during eras of structure formation, they are "hot" (warm) dark matter

Density inhomogeneities: Growth of structure using linear perturbation theory

So far we assumed that the matter, radiation field, i.e., all physical quantities in our models of the Universe are distributed in a homogeneous manner. But the real universe contains inhomogeneous structures like galaxies, clusters of galaxies, clusters of clusters (superclusters), etc. Let's discuss mechanism for formation of these density structures.

The formalism we will use is linear perturbation theory. We shall assume that somehow during inflation, quantum fluctuations in the inflaton field ϕ led to the formation energy density fluctuations. These ultimately turn into fluctuations in matter (i.e., dark matter and baryons) and radiation (i.e., photons). As we shall see the density fluctuations grow in time and ultimately generate variations in the CMB temperature such that at decoupling these are very small. Thus, the departure from uniform density were very small so we can safely use linear perturbation theory. But ultimately, later on, non-linear theory must be developed to describe evolution of small-scale perturbations which got non-linear first.

Newtonian vs Relativistic Description

Correct description involves perturbing the Einstein field equations by taking perturbations of the FRW metric & stress energy tensor. i.e.,

$$g_{ab} = \underset{\text{FRW}}{\bar{g}_{ab}} + \underset{\text{perturbation}}{h_{ab}} \quad \text{where } |h_{ab}| \ll \bar{g}_{ab}$$

$$T_{ab} = \underset{\substack{\text{smooth}\\ \text{fluid}}}{\bar{T}_{ab}} + \underset{\substack{\text{density and}\\ \text{pressure perturbation}}}{\delta T_{ab}}$$

we wind up with eqs.:

$$\hat{\mathcal{L}}(g_{ab})\, h_{ab} = \delta T_{ab}$$

2nd order total DES where $\hat{\mathcal{L}}$ is a linear differential operator (2nd order) that depends on the background metric. In fact ~~we shall~~ usual method is to take Fourier transform and and we can solve eq. for given wave numbers, i.e. wave lengths, λ

But shall take simpler approach

- Modes $\lambda < l_H$ horizon radius, GR effects due to curvature are negligible and we can use Newtonian theory

- But as we saw earlier, ~~almost~~ all modes were initially outside FRW

Horizon. So Newtonian theory ~~as~~ is useful only after mode λ enters horizon at time $t_{enter}(\lambda)$. Earlier evolution: GR.

<u>Two step procedure</u>

$\rho = \rho_0 + \delta\rho$ at given λ

(a) ($t < t_{enter}(\lambda)$) $\lambda > l_H$ ∴ <u>GR</u> needed to evolve $\delta\rho_\lambda(t)$ from some initial time (post inflation) to $t_{enter}(\lambda)$

(b) ($t > t_{enter}(\lambda)$)

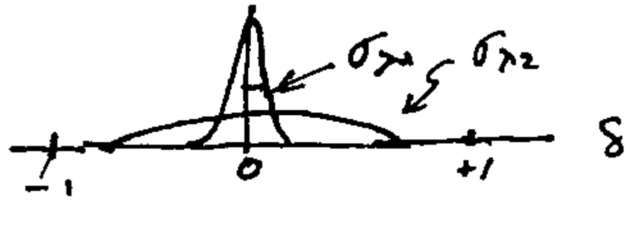
$\lambda < l_H$ ∴ Newton theory ok for evolution.

$\lambda > l_H$ ∴ GR still necessary

<u>Initial Conditions</u>

~~Infact~~ We shall assume that $\delta\rho(\lambda)$ ~~is~~ are described by a Gaussian random field.

Let $\delta \equiv \dfrac{\delta\rho}{\rho_0} = \dfrac{\rho - \rho_0}{\rho_0}$



Dispersion of distribution at each λ

we shall see that for $\lambda_2 < \lambda_1$, $\sigma_{\lambda_2} > \sigma_{\lambda_1}$

Power spectrum describes functional form of σ_λ or σ_k where k is (inflation

wave-number: $P(k) = A \cdot k^n$, $n \simeq 1$)

So shorter wavelength perturbations (higher k) start out, on the average, with larger density amplitudes (more later)

Assumptions

~~Smooth~~

Smooth Universe: Three main components

Baryons: ρ_B

Dark-Matter: ρ_{DM}

Relativistic Matter: ρ_R

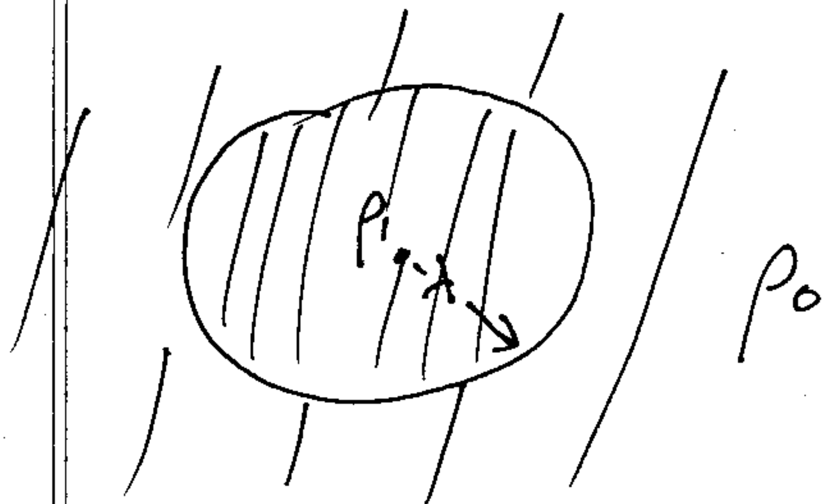
Dark Matter: Decouples at some temperature T_D and becomes non-relativistic at temperature $T = T_{NR} = m_x c^2/k_B$. In all cases $T_{NR} > T_{eq}$ (epoch of matter/radiation equilibrium)

~~Equation of State~~:

Simple Description

Consider perturbation with $\lambda > l_H$: superhorizon perturbation. Take a spherical region with radius λ with mean density ρ_1 embedded in a $k=0$ Friedmann background model with density ρ_0.

So $\rho_1 = \rho_0 + \delta\rho$ where $|\delta\rho| \ll \rho_0$



Spherical Symmetry (Birkhoff's GR theorem) tells us that inner region not affected by matter outside. Thus inner region evolves like a $k=+1$ Friedmann Universe:

Inside
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{c^2}{a^2}$$

$$H_1^{\,2} = \frac{8\pi G\rho_1}{3} - \frac{c^2}{a_1^{\,2}} \qquad : H_1 = \frac{\dot{a}_1}{a_1}$$

Outside Background
$$H_0^{\,2} = \frac{8\pi G\rho_0}{3} \qquad : H_0 = \frac{\dot{a}_0}{a_0}$$

Compare perturbed universe with background Universe when their expansion rates are equal that is when $H_1 = H_0$.

In that case:
$$\frac{8\pi G\rho_1}{3} - \frac{c^2}{a_1^{\,2}} = \frac{8\pi G\rho_0}{3}$$

$$\frac{8\pi G}{3}(\rho_1 - \rho_0) = \frac{c^2}{a_1^{\,2}}$$

Therefore
$$\frac{\rho_1 - \rho_0}{\rho_0} = \frac{\delta\rho}{\rho} = \frac{3c^2}{8\pi G\rho_0 a_1^{\,2}}$$

In general if $H_1 = H_0$ at some time, then $a_1 \neq a_0$ at that time. But if $\frac{\delta\rho}{\rho} \ll 1$, a_1 will differ from a_0 by a small amount. So let's set $a_1 \simeq a_0$ on RHS of last equation. ~~the~~ In that case we have

$$\boxed{\frac{\delta\rho}{\rho_0} \simeq \frac{3c^2}{8\pi G(\rho_0 a_0^2)}}$$

Since $\rho_0 \propto a_0^{-4}$ in RD phase at $t < t_{eq}$
$\rho_0 \propto a_0^{-3}$ in MD phase at $t > t_{eq}$

$$\frac{\delta\rho}{\rho} \propto \begin{cases} a^2 \; ; & t < t_{eq} \\ a \; ; & t > t_{eq} \end{cases}$$

This tells us how mode with $\lambda > l_H$ grows