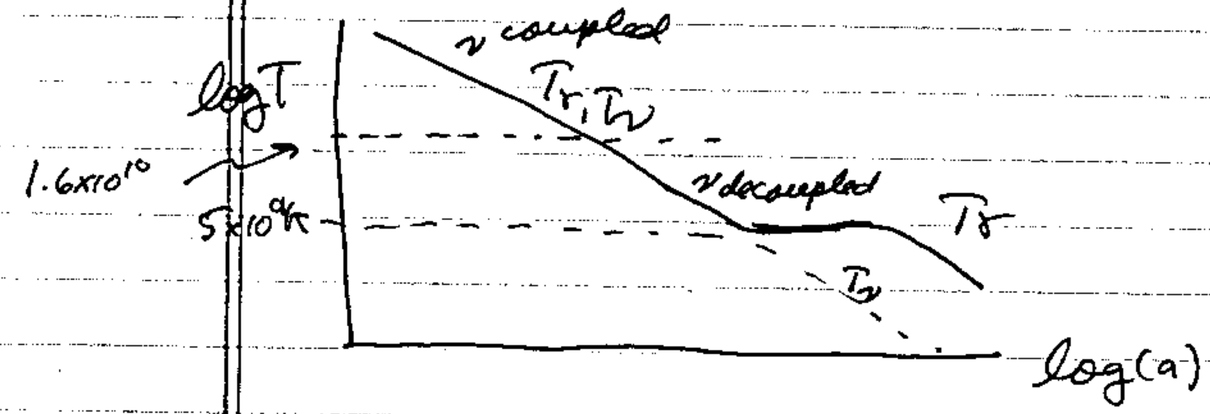
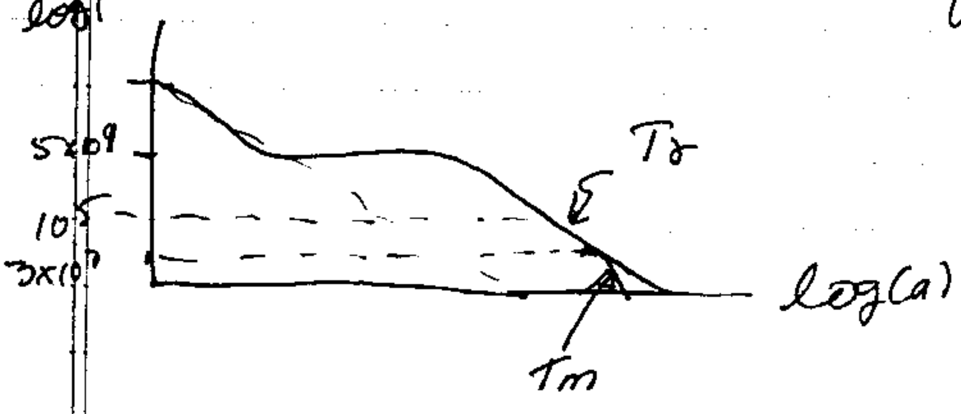
Recap

(1) Reheating at $T \simeq 5\times10^9 K$



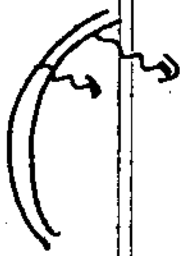
when $e^+ + e^-$ pairs annihilate, photon gas gets heated. But T_γ doesn't increase. Why? Because ~~at about this temperature~~ adiabatic cooling due to expansion balances heat input due to expansion. Result is that $T \simeq$ constant until e^+, e^- annihilation is finished. At that time, Then T_γ again falls off like $1/a$ due to expansion, but at a higher value than T_ν

(2) Recombination and Surface of last scattering



We began discussing recombination of the ionized plasma into neutral gas & the decoupling of CMB photons from matter. Without free e^-, radiation doesn't interact with matter since ⓐ no free e^- for Compton or Thomson scattering, ⓑ photons not sufficiently energetic to ionize H^0.

- Important because
  - from what z do CMB propagate to us?
  - width of last scattering surface limits amplitude of Temperature variation below critical angular scales. Photons arriving from different T layers will smear out angular variations due to density inhomogeneities, etc.



<u>Saha Equation:</u>

In thermodynamic equilibrium, one uses Saha equation. We found that if (Note x also $= \frac{n_e}{n}$): $x = \frac{n(H^+)}{n}$, we can trace fractional ionization of hydrogen as a function of redshift.

Since $1-x = n(H^0)/n$ is neutral fraction.

$$\frac{x^2}{1-x} = \frac{(2\pi m_e k_B T)^{3/2}}{n h^3} \exp\left[-\frac{\chi_H}{k_B T}\right]$$

Baryon density $n(z) = n(0)(1+z)^3 = n_0 (T/2.728)^3 n(0)$

where current $n(0) = n_B(0) = 2.25 \times 10^{-7} \left(\frac{\Omega_b h^2}{.02}\right) \text{cm}^{-3}$

Put all the numbers in we get

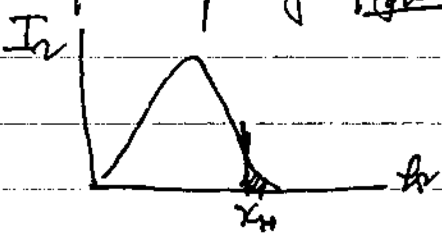
$$\frac{x^2}{1-x} = \frac{2.23\times10^{23}}{\left(\frac{\Omega_b h^2}{.02}\right)T^{3/2}} \exp\left[-\frac{158{,}000}{T}\right]$$

where $\chi_H/k_B = 158{,}000$: Solution for x shown in the figure.

Comments:

(i) Recombination epoch:

Cosmological ~~y~~plasma stays ionized as we move forward in time until $z \simeq 1370$ where $x \simeq 0.5$. Notice $T \simeq 3750$ K. At first this might seem surprising because ionization potential of hydrogen $\chi_H/k = 158{,}000$. So the implication is BB ~~photons~~ spectrum with peak ~~energy~~ much less than ionization potential ~~are~~ is capable of keeping ~~[illegible]~~ H ionized



The reason for this is the large entropy per baryon of the radiation field; i.e., there are many more free-states for the ~~[illegible]~~ e^- to occupy than the bound states, which is the reason for the large coefficient in above equation

Recall Entropy per baryon: $\sigma = \frac{4}{3}\frac{a_B T^3}{n_B k_B}$

or: $\sigma = \frac{4}{3}\frac{T^3}{n_B k_B} \times \left(\frac{8\pi^5 k_B^4}{15 h^3 c^3}\right)$

$$\therefore \quad \frac{1}{n_B h^3} = \frac{c^3}{k_B^3 T^3} \times \frac{45}{32\pi^5}\,\sigma$$

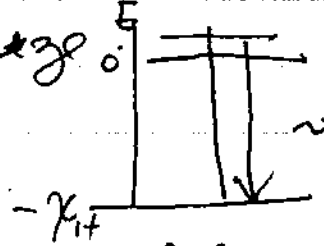
From Saha equation $\frac{x^2}{1-x} = (2\pi)^{3/2}\frac{m_e^{3/2} k_B^{3/2} T^{3/2}}{n h^3} \exp\left(-\frac{\chi_H}{k_B T}\right)$

we have $$\frac{x^2}{1-x} = (2\pi)^{3/2} m_e^{3/2} k_B^{3/2} T^{3/2} \times \frac{c^3}{k_B T^3} \times \frac{45}{32\pi^5}\,\sigma \exp\left(-\frac{\chi_H}{k_B T}\right)$$
$$= \left(\frac{(2\pi)^{3/2}\times 45}{32\pi^5}\right) \times \left(\frac{m_e c^2}{k_B T}\right)^{3/2} \sigma \cdot e^{-\left(\frac{\chi_H}{k_B T}\right)}$$

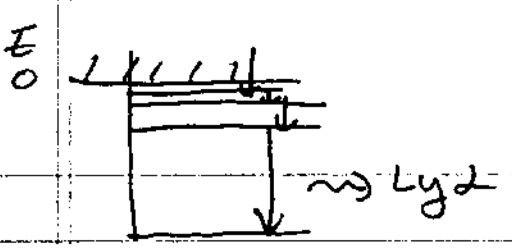
So high value of $\sigma \simeq 2\times 10^9$ plus $m_e c^2/kT \gg 1 \Rightarrow$ gas is highly ionized until T gets very low $\ll \chi_H/k_B$

(2) Non-Equilibrium Reality includes

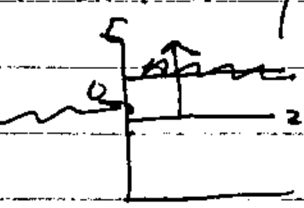
Saha equation ~~assumes~~ recombinations of e^- to ground state that result in net production of neutral H^0 atoms. But this produces photon with $h\nu > \chi_H$ which travel until they encounter neutral atom, which they proceed to ionize:



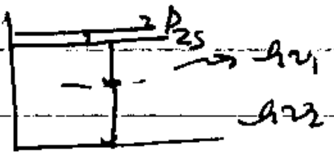
$\rightsquigarrow h\nu > \chi_H$. So have to wait for smaller jumps to higher energy excited states. But these inevitably result in Ly α production.



But Ly α is trapped and causes presence of many atoms in excited 2p state, which result in reionization.

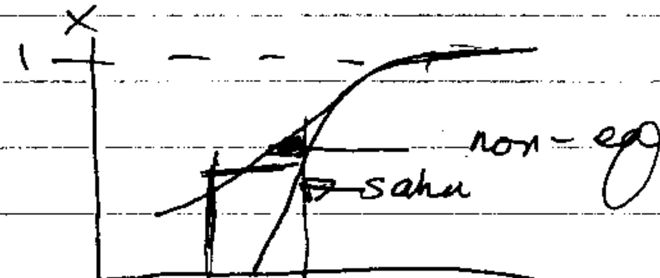


The only way out is for collisional de-excitation from 2p → 2s. 2s → 1s results in forbidden 2 photon emission, which transfers ionization energy into photons with $h\nu < h\nu_{Ly\alpha}$. All of these processes suppress production of H^0 atoms and slower recombination occurs.



In this case one solves a time-dependent rate equation: something like:

$$\frac{d}{dt}(n_e a^3) = -\alpha n_e^2 a^3 + \beta n_{2s} a^3$$



where Saha eq. only depended on $\Omega_b h^2$ through baryon density n, in more accurate case Ω_m comes in from converting time derivative to redshift derivative. So near $z \simeq 1000$ one finds

$$X(z) \simeq 2.4\times10^{-3} \frac{(\Omega_m h^2)^{1/2}}{\Omega_b h^2}\left(\frac{z}{1000}\right)^{12.75}$$

Optical depth to Thomson scattering by residual free electron

(3) $\tau = \int n_e \sigma_{Thomp}\, c\, dt = \sigma_{Thomp} \int X n_B c \frac{dt}{dz} dz$

But $\left|\frac{dt}{dz}\right| \propto \frac{1}{(\Omega_m h^2)^{1/2}}$;

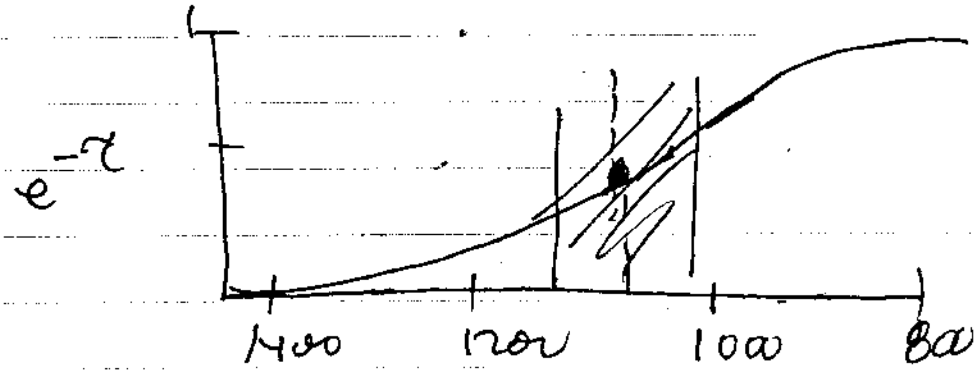
$\therefore\ X n_B \frac{dt}{dz} \propto \frac{(\Omega_m h^2)^{1/2}}{(\Omega_b h^2)} \times \Omega_b h^2 \times \frac{f(z)}{(\Omega_m h^2)^{1/2}}$

$\therefore$ Thomson τ <u>independent of cosmological parameters</u>.

One finds.. $\tau(z) = .37 \left(\frac{z}{1000}\right)^{14.25}$

Since $I_\nu = I_{\nu 0}(e^{-\tau}) \Rightarrow e^{-\tau}$ = survival fraction.

$e^{-\tau} = 0.5$ when $z \simeq 1090$!



width from $e^{-\tau} = .25$ to $e^{-\tau} = .75$; $\sigma_z \simeq 90$

Radial width: $c \frac{dt}{dz} \sigma_z$

This will produce random temperature variations with length scale

$$d \simeq c \left|\frac{dt}{dz}\right| \sigma_z$$

Evaluate : $d \simeq c\,\sigma_z \left(+ \frac{1}{(1+z)\,H(z)} \right)$

$$d \simeq c\,\sigma_z \times \frac{1}{(1+z)\, H_0\, \Omega_m^{1/2} (1+z)^{3/2}}$$

Therefore $\quad d \simeq \frac{c}{H_0} \times \frac{\sigma_z}{\Omega_m^{1/2} (1+z)^{5/2}}$

Put in numbers:

$z \simeq 1090,\ \sigma_z \simeq 90,\ \Omega_m \simeq 0.3,\ c H_0^{-1} \simeq \frac{3000}{h}$ Mpc

We find: $d = \frac{1.3 \times 10^{-2}}{h}$ Mpc

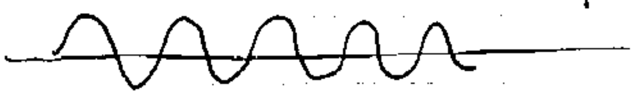
Assume transverse length scale $\simeq d$ we predict angular variations

$$\theta_d \simeq \frac{d}{d_A}$$

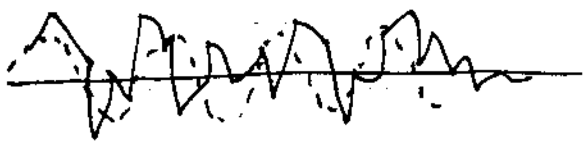
Since $d_A = \frac{9.1}{h}$ Mpc, $\theta_d \simeq 4.5'$

Suppose we are searching for intrinsic temperature variations with scales $< d$ back to precise decoupling redshift

Theoretical: $\frac{\delta T}{T}$



With width smearing



As a result, ~~[illegible]~~ temperature variations on scales $\Theta < \Theta_d$ will be smeared out and difficult to observe: more later.

Light Element Synthesis

Let's take a look at nucleons, which are (a) non-relativistic and (b) do not contribute significantly to energy density and thus have no influence on dynamics: they are along for the ride.

Production of ${}_1H^2$, ${}_2He^4$: major steps

(1) $T > 10^{10}$ K ($t < 1$ sec): (n, p) in TE with γ, e^-, and indirectly with each other.

(2) $T \simeq 10^{10}$ K: (n, p) equilibrium is broken. (n/p) ratio "freezes out" at $(n/p) \simeq 0.1$

(3) $T \simeq 10^9$ K: all remaining neutrons react with protons to form ${}_1H^2$, which burns almost completely to ${}_2He^4$: competes with $n \to p$ decay

(A) (n/p) Ratio

Weak interactions couple n, p with leptons and each other

$$\left\{\begin{array}{l} n + \nu_e \rightleftarrows p + e^- \\ n + e^+ \rightleftarrows p + \tilde{\nu}_e \\ n \to p + e^- + \tilde{\nu}_e \end{array}\right.$$

ν's are crucial!

Define:

(i) Neutron fraction: $X_n = \frac{n}{n+p}$

(ii) Proton fraction: $X_p = \frac{p}{n+p} = 1 - X_n$

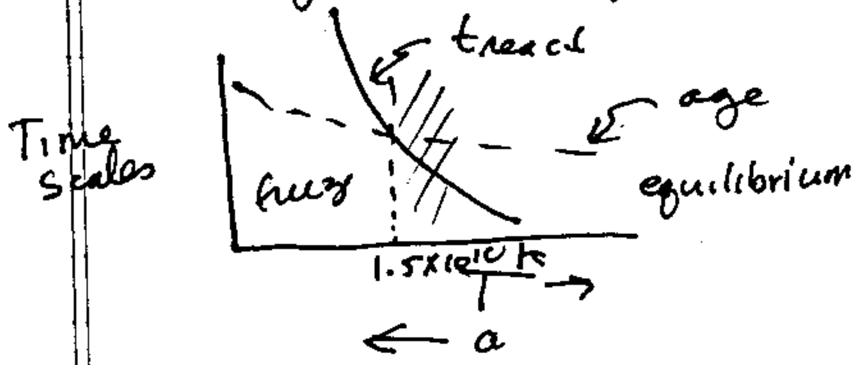
Reaction Rate: $\frac{dX_n}{dt} = X_p \langle v\sigma \rangle_{p \to n}^{n_\nu} - X_n \langle \sigma v \rangle_{n \to p}^{n_\nu}$

~~But $\langle \sigma v \rangle$ are mediated by ν's and depend on ν density.~~

Recall: $n_\nu = 25 T^3 \ \mathrm{cm}^{-3}$, $v_\nu \simeq c$, $\sigma \propto T^2$

Therefore: $t_{react} \simeq \frac{1}{\langle \sigma v \rangle n_\nu} \propto \frac{1}{T^5}$

But age $t \propto 1/T^2$



So for $T > 1.5\times10^{10}$ K, $t_{react} < t$, and n, p are in TE

So that

$$\frac{dX_n}{dt} = 0 \implies$$

$$(1 - X_n)\langle \sigma v \rangle_{p \to n} = X_n \langle \sigma v \rangle_{n \to p}$$

$$X_n \left[\langle \sigma v \rangle_{n \to p} + \langle \sigma v \rangle_{p \to n} \right] = \langle \sigma v \rangle_{p \to n}$$

$$X_n = \frac{\langle \sigma v \rangle_{p\to n}}{\langle \sigma v \rangle_{n\to p} + \langle \sigma v \rangle_{p\to n}}$$

$$X_n = \frac{1}{1 + \frac{\langle \sigma v \rangle_{n\to p}}{\langle \sigma v \rangle_{p\to n}}}$$

For non-relativistic nucleons: $\frac{\langle \sigma v \rangle_{n\to p}}{\langle \sigma v \rangle_{p\to n}} = \exp\left[\frac{(m_n - m_p)c^2}{k_B T}\right]$

But $X_n = \frac{n}{n+p} = \frac{1}{1 + \frac{p}{n}}$

Therefore $\boxed{\frac{n}{p} = \exp\left[- \frac{(m_n - m_p)c^2}{k_B T}\right]}$

Note: $(m_n - m_p)c^2 = 1.293$ MeV

$$\frac{(m_n - m_p)c^2}{k_B} = 1.5 \times 10^{10}\ K$$

As a result: $X_n = \frac{1}{1 + \exp\left(\frac{1.5\times 10^{10}}{T}\right)}$

Consequences:

(a) $T \gg 1.5\times 10^{10}$; $X_n \to 0.5$

(b) As $T \to 1.5\times 10^{10}$ K: equilibrium stops and X_n freezes out at critical $X_n(T_{crit})$

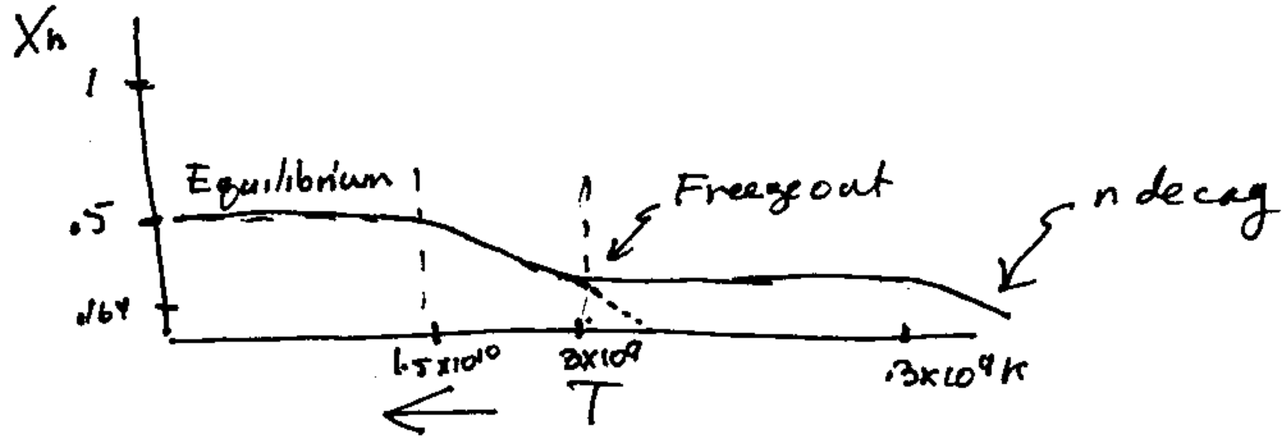
(c) Accurate calculations show..

$$T_{crit} \simeq 3\times10^9 K, \text{ where } X_n = 0.164$$

Freeze Out:
at $T < T_{crit}$, all interactions with leptons effectively cease. But free n decay still occurs:

$$X_n = 0.164 \exp\left(-\frac{t}{1013\,sec}\right)$$

Schematic Figure



n decay important at $T < 0.3\times10^9 K$
why?

$$t = 10^{20}/T^2 \Rightarrow T = \frac{10^{10}}{\sqrt{t}}$$

For neutron lifetime $t = t_n = 1013\,sec$

$$T = \frac{10^{10}}{\sqrt{1013}} = 0.3\times10^9 K$$

But this ignores fusion to $_1H^2$ and $_2He^4$

That is fusion reactions compete with free n decay!.

Step (1): $n + {}_1H^1 \rightleftarrows {}_1H^2 + \gamma$

(note this is much faster than ${}_1H^2$ production in stellar interiors where no free n's are present. There we must rely on much slower weak interaction to convert $p \to n$)

$\{{}_1H^1 + {}_1H^1 \to {}_1H^2 + e^+ + \nu_e\}$

at $T \sim 10^9 K$ t_{react}(strong) $\ll t$ for these reactions so equilibrium holds for element production even though t_{react}(weak) $\gg t$ at this temperature

Relevant Reactions { analogy with the recombination and use of Saha Equation }

Recomb.: $e^- + H^+ \rightleftarrows H^0 + \gamma$

$\uparrow \quad \uparrow \quad \uparrow$

Fusion: $n + {}_1H^1 \rightleftarrows {}_1H^2 + \gamma$

Partition functions:

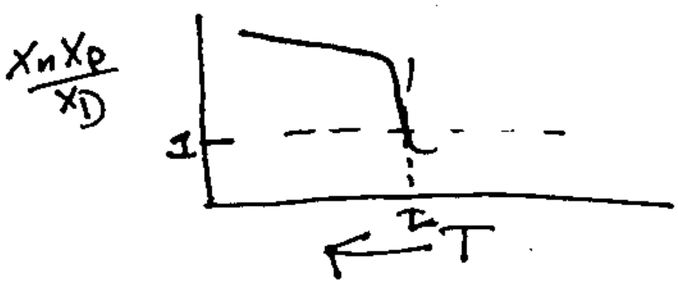
$$Z_n = 2\,(2\pi m_n k_B T)^{3/2}\, V/h^3 \quad \text{(free particle)}$$
$$Z_p = 2\,(2\pi m_p k_B T)^{3/2}\, V/h^3 \quad \text{"}$$
$$Z_{{}_1H^2} = \underset{\uparrow\,\text{spin }1}{3}\,(2\pi m_D k_B T)^{3/2}\,\frac{V}{h^3}\,\exp(B_D/k_B T)$$

$$B_D = 2.225\ \text{MeV} \qquad \frac{B_D}{k_B} = 2.6\times10^{10}\ K$$

As a result:

$$\frac{X_n X_p}{X_D} = \frac{Z_n Z_p}{Z_D} = \frac{4}{3}\frac{(2\pi k_B T)^{3/2}}{h^3 n}\left(\frac{m_n m_p}{m_D}\right)^{3/2} e^{-\frac{B_D}{k_B T}}$$

Consequently TE shifts from $\frac{X_n X_p}{X_D} \gg 1$ at high T to $\frac{X_n X_p}{X_D} = 1$ at critical T



$T_{9*} = 0.71$

$\Omega_B h^2 = .02$

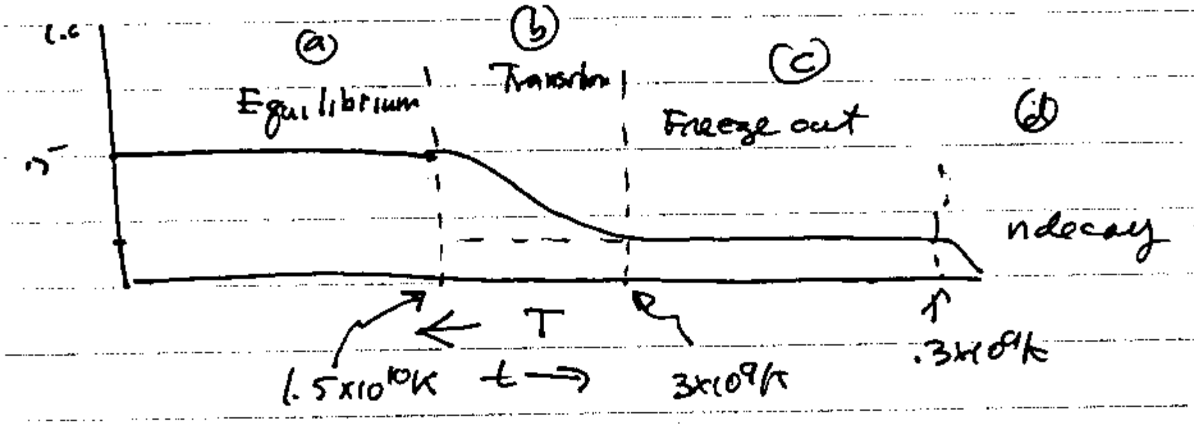
Deuteron Burning begins at $(T_9)_*$

~~Production $n + {}_1H^1 \rightleftarrows {}_1H^2 + \gamma$ — ${}_1H^3 + {}_1H^1$ strong interactions~~

~~Other reactions ${}_1H^1 + {}_1H^2 \to {}_2He$ EM interactions~~

Recap: Light Element synthesis

(1) Neutron fraction



4 critical regimes

$X_n = \dfrac{1}{1+\exp\left(\frac{1.5\times10^{10}}{T}\right)}$

(a) $T \gg 1.5\times10^{10}$ n/p in equilibrium $\{t_{react} \ll t\}$
$X_n = 0.5$

(b) transition $t \sim t_{react}$: $3\times10^9 < T < 1.5\times10^{10}$

(c) Freezeout $t_{react} \gg t$ $X_n = 0.164$

(d) n-decay : $X_n = 0.164 \exp\left(-\frac{t}{t_n}\right)$
$T < .3\times10^9 K$

(2) Fusion Reactions

$$n + {}_1H^1 \rightarrow {}_1H^2 + \gamma$$

Saha equation:

$$\frac{X_n X_p}{X_D} = \frac{4}{3}\frac{(2\pi k_B T)^{3/2}}{n h^3}\left(\frac{m_n m_p}{m_D}\right)^{3/2} \exp\left(-\frac{B_D}{k_B T}\right)$$

T.E. shifts from $\frac{X_n X_p}{X_D} \gg 1$ at high T

to $\frac{X_n X_p}{X_D} = 1$ at

lower temperatures.

at $T \simeq 3\times10^9 K$, ${}_2He^4$ could be produced by reactions:

$${}_1H^2 + {}_1H^2 \rightarrow {}_2He^4 + \gamma$$
$${}_1H^1 + {}_1H^2 \rightarrow {}_2He^3 + \gamma$$
$${}_1H^1 + n \rightarrow {}_1H^3 + \gamma$$
$${}_1H^1 + {}_1H^3 \rightarrow {}_2He^4 + \gamma$$
$$n + {}_2He^3 + n \rightarrow {}_2He^4 + \gamma$$

$T \sim 3\times10^9 K$ But these are based on abundances of ${}_1H^2$, ${}_2He^3$, and ${}_1H^3$. But these reactions do not occur because of ~~[illegible]~~ mass fractions X_D, X_{He^3}, X_{H^3} are still very small (10^{-12}, 10^{-19}, 5×10^{-19} respectively). even down to $\sim 3\times10^9 K$.

$T \simeq 1\times10^9 K$ Only at $T \leq 1\times10^9 K$ are the X_A's sufficiently large for reactions to go fast enough to produce an equilibrium abundance of ${}_2He^4$. At these temperatures most of the ~~nucleons~~ neutrons wind up in strongly bound ${}_2He^4$ nuclei: ~~[illegible]~~ so, At end ~~[illegible]~~ of free n decay, dominant species are:

$${}_1H^1 \quad , {}_2He^4$$

${}_2He^4$ Abundance

Consider a box with N nucleons

$X_n N$ = number of neutrons

$(1-X_n)N$ = " " protons

$\frac{X_n N}{2}$ = " " ${}_2He^4$ nuclei $\left\{ \begin{array}{l} 2n \text{ to} \\ \text{every} \\ {}_2He^4 \text{ nucleus} \end{array} \right.$

~~[illegible]~~ ~~[illegible]~~

Masses: $M_{TOT} = m_n N$

$$M({}_2He^4) = (4m_n)\left(\frac{X_n N}{2}\right) = 2 m_n X_n N$$

Abundance by mass: $Y = \dfrac{M({}_2He^4)}{M_{TOT}} = \dfrac{2\cancel{m_n} X_n \cancel{N}}{\cancel{m_n}\cancel{N}}$

$$\boxed{Y = 2X_n}$$

$t_N \simeq \dfrac{10^{20}}{T^2}$

at $T_N \simeq 1\times10^9 K$, $t_N = \dfrac{10^{20}}{10^{18}} \sim 100\,sec$

more accurate $t_N \simeq 180\,sec$

$\therefore\; X_n = 0.164 \times \exp\left(-\dfrac{180}{1013}\right) \simeq 0.13$

$\Rightarrow \boxed{Y_n \simeq 0.26}$ ← pretty close to obs.

Deuterium Diagnostic

As ${}_2He^4$ builds up, it does so at expense of ${}_1H^2$ and ${}_1H^3$. Reaction rates depleting ${}_1H^2$ is given by

$$\Gamma \propto X_D n_B \langle \sigma v\rangle$$

In fact $\Gamma = 1.9\times10^7 (T/10^{10}K)^3 (\Omega_B h^2) X_D\, s^{-1}$
(cross-sections ~ $1/v$)
This freezes out when $\Gamma = H(t) = 1/2t = .28(T/10^{10})^2 s^{-1}$
So X_d freezes out when $T \sim 10^9 K$ and

$$\boxed{X_d \simeq 1.2\times10^{-7}/\Omega_b h^2}$$

Ω_B-dependence ~~[illegible]~~ abundance of Deuterium is a valuable diagnostic for determining Ω_B

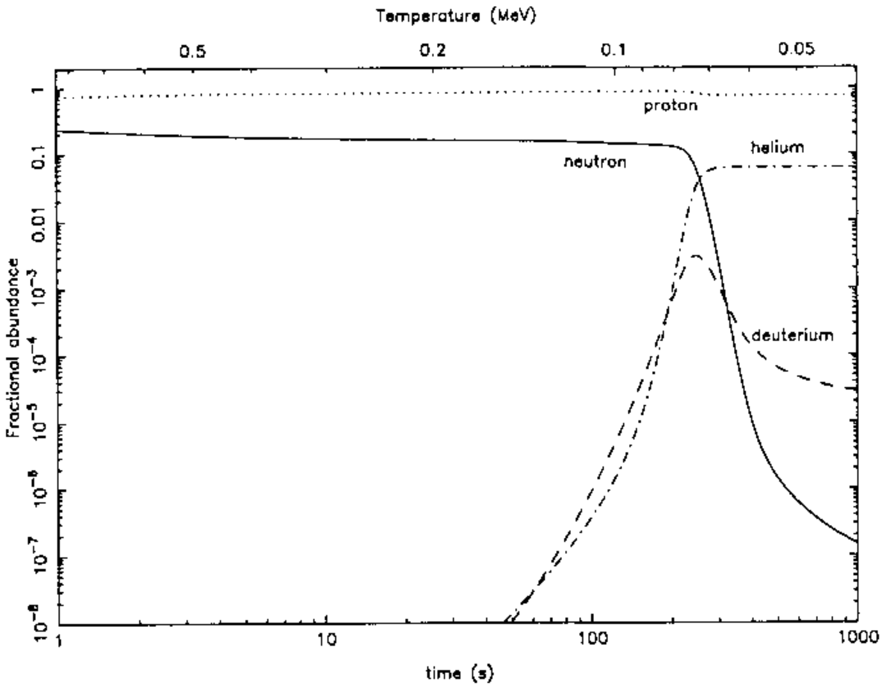


FIG. 1: The evolution of mass fraction of different species during nucleosynthesis

Using $n_1 = n_n, n_3 = n_p$ and $n_2, n_4 = n_l$ where the subscript l stands for the leptons, Eq. (23) becomes

$$\frac{1}{a^3}\frac{d(n_n a^3)}{dt} = \mu n_l^{\rm eq}\left(\frac{n_p n_n^{\rm eq}}{n_p^{\rm eq}} - n_n\right). \tag{26}$$

We now use Eq. (24), write $(n_l^{\rm eq}\mu) = \lambda_{np}$ which is the rate for neutron to proton conversion and introduce the fractional abundance $X_n = n_n/(n_n + n_p)$. Simple manipulation then leads to the equation

$$\frac{dX_n}{dt} = \lambda_{np}\left((1 - X_n)e^{-Q/T} - X_n\right). \tag{27}$$

Converting from the variable t to the variable $s = (Q/T)$ and using $(d/dt) = -HT(d/dT)$, the equations we need to solve reduce to

$$-Hs\frac{dX_n}{ds} = \lambda_{np}\left((1 - X_n)e^{-s} - X_n\right); \quad H = (1.1\ {\rm sec}^{-1})\ s^{-2}; \quad \lambda_{np} = \frac{0.29\ {\rm s}^{-1}}{s^5}\left[s^2 + 6s + 12\right]. \tag{28}$$

It is now straightforward to integrate these equations numerically and determine how the neutron abundance changes with time. The neutron fraction falls out of equilibrium when temperatures drop below 1 MeV and it freezes to about 0.15 at temperatures below 0.5 MeV.

As the temperature decreases further, the neutron decay with a half life of $\tau_n \approx 886.7$ sec (which is *not* included in the above analysis) becomes important and starts depleting the neutron number density. The only way neutrons can survive is through the synthesis of light elements. As the temperature falls further to $T = T_{\rm He} \simeq 0.28$MeV, significant amount of He could have been produced if the nuclear reaction rates were high enough. The possible reactions which produces ^{4}He are $[D(D,n)\ ^3He(D,p)\ ^4He,\ D(D,p)\ ^3H(D,n)\ ^4He,\ D(D,\gamma)\ ^4He]$. These are all based on D, ^3He and ^3H and do not occur rapidly enough because the mass fraction of D, ^3He and ^3H are still quite small $[10^{-12}, 10^{-19}$ and 5×10^{-19} respectively$]$ at $T \simeq 0.3$MeV. The reactions $n + p \rightleftharpoons d + \gamma$ will lead to an equilibrium abundance ratio of deuterium given by

$$\frac{n_p n_n}{n_d n} = \frac{4}{3}\left(\frac{m_p m_n}{m_d}\right)^{3/2}\frac{(2\pi k_B T)^{3/2}}{(2\pi\hbar)^3 n}e^{-B/k_B T} = \exp\left[25.82 - \ln\Omega_B h^2 T_{10}^{3/2} - \left(\frac{2.58}{T_{10}}\right)\right]. \tag{29}$$

The equilibrium deuterium abundance passes through unity (for $\Omega_B h^2 = 0.02$) at the temperature of about 0.07 MeV which is when the nucleosynthesis can really begin.

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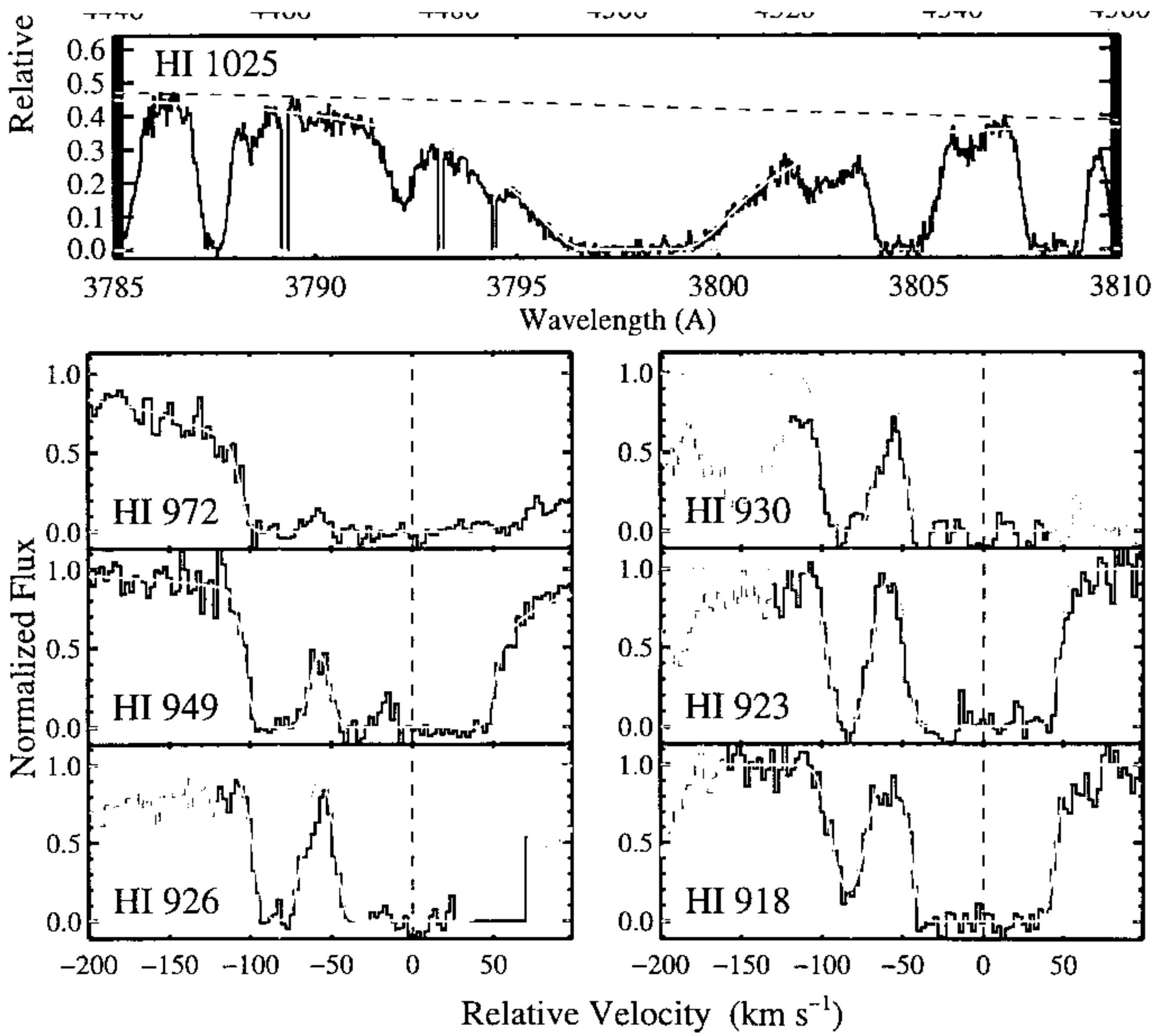

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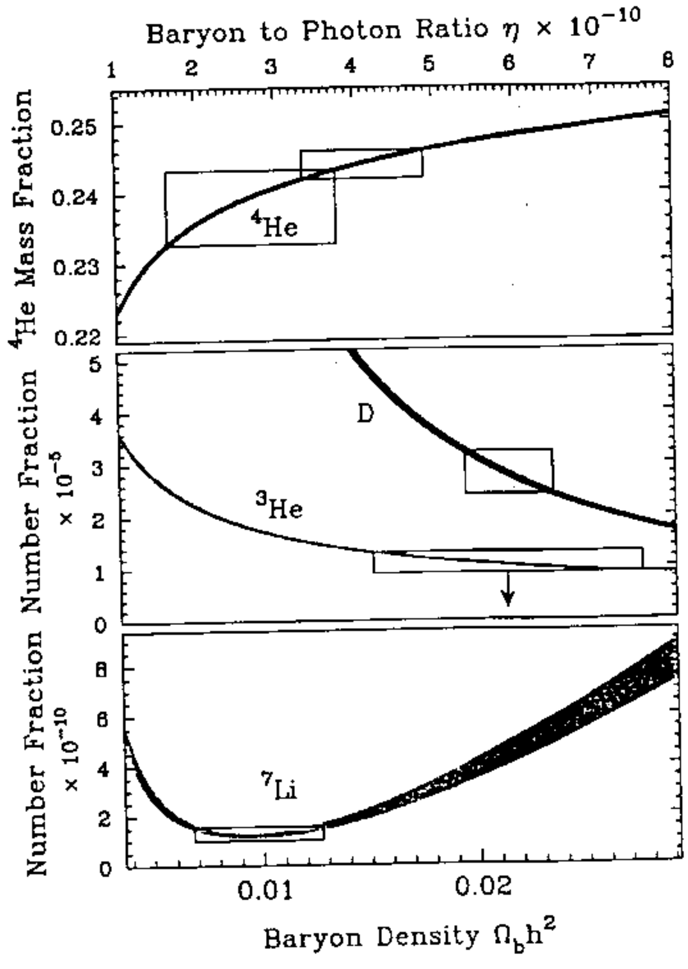


Baryon to Photon Ratio $\eta \times 10^{-10}$
1
2
3
4
5
6
7
8
^{4}He Mass Fraction
0.25
0.24
0.23
0.22
^{4}He
Number Fraction $\times 10^{-5}$
5
4
3
2
1
0
D
^{3}He
Number Fraction $\times 10^{-10}$
8
6
4
2
0
^{7}Li
0.01
0.02
Baryon Density $\Omega_b h^2$

Figure shows build-up then destruction of $_1H^2$
also shown " " of $_2He^4$

The best fit of the D/H ratio in the figures comes from measuring D absorption lines at high redshift. Because of the isotope shift predicted in atomic absorption lines, the Rydberg constant, which depends on reduced mass causes an isotope shift of 80 km s^{-1} in Lyα and higher order Lyman series transitions that Tytler and colleagues managed to measure. They also had to measure Ly lines to get H

Quasar absorption: $\Omega_B h^2 = 0.024 \pm 0.0020$

Results

Clearly, Ω_B is significantly smaller than Ω_m, ~~total~~ density contributed by all forms of non-relativistic matter, which SNIa results give as

$$\Omega_m \simeq 0.3$$

CMB measurements, As we shall see CMB measurements of temperature anisotropies result in

CMB

$$\Omega_B h^2 = 0.0223 \pm .0008$$
$$\Omega_M h^2 = .127^{+.007}_{-.013}$$

As a result: $\boxed{l_H(t) = \frac{c}{H(t)}}$

(c) Ratio of distance to F.O. to Horizon Radius

$$\mathcal{R} = \frac{l_{FO}(t)}{l_H(t)} = \frac{a(t)\chi}{cH^{-1}} = \frac{a(t)H(t)}{c}\chi$$

$$\mathcal{R} = \frac{\dot{a}\chi}{c}$$

(d) Implications:

Second Friedmann Equation: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$

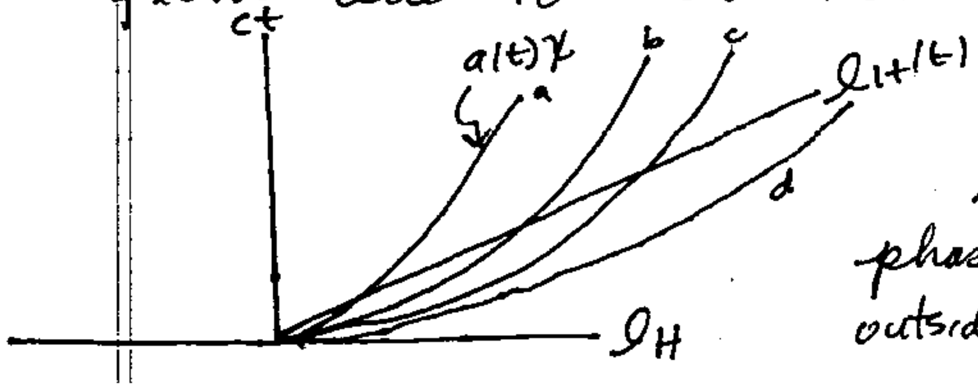
Since $\rho > 0$; $P > 0$ during radiation-dominated

$\ddot{a} < 0 \Rightarrow$ Universe decelerates

Stated differently $\dot{a}$ decreases with time

Consequently $\mathcal{R} = \frac{l_{FO}(t)}{l_H(t)}$ decreases with t

Result: Horizon grows more rapidly than scale factor due to deceleration of the Universe



Just as in matter-dominated phase, objects initially outside horizon enter later on

Thus total mass density of matter $\simeq 5 \to 6$ times larger than mass density of ordinary baryonic matter. What is the nature of the dominant form; i.e., dark matter?

Inflation

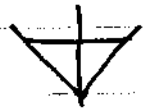
(I) Horizon Problem

Consider time-dependence of horizon proper radius during radiation-dominated phase:

$$ds^2 = c^2 dt^2 - a^2(d\chi^2 + F^2(\chi)d\Omega^2)$$

Lightlike Geodesics: $c^2 dt^2 = a^2 d\chi^2$

(a) Forward lightcone:



$$c\,dt = + a(t)d\chi$$

Comoving coordinate: $\chi(t) = \int_0^t \frac{c\,dt}{a(t)}$

proper radius: $l_H(t) = a(t)\chi(t)$

$$\rightarrow l_H(t) = a(t)\int_0^t \frac{c\,dt'}{a(t')}$$

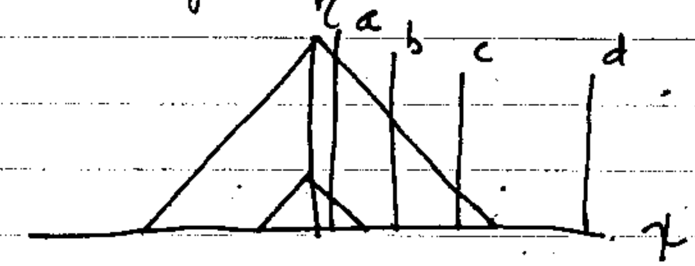
Since $a(t) = (\text{const})\, t^{1/2}$ during radiation phase

$$\boxed{l_H(t) = 2ct}$$

(b) Hubble Parameter

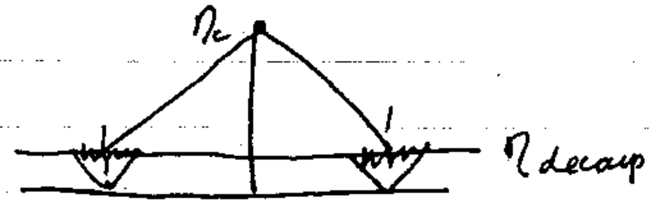
$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t}$$

So if at time t galaxy d is outside horizon, it had <u>always</u> been outside horizon!

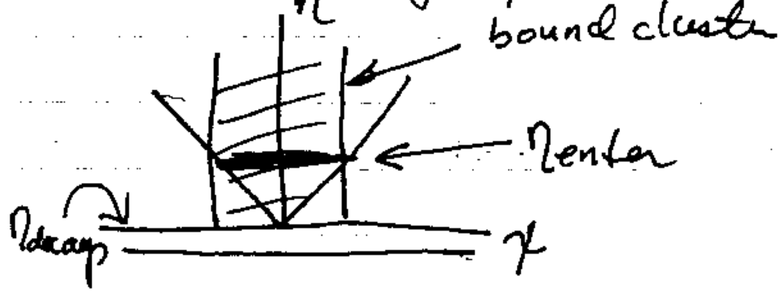


(II) Physical Problems Raised by Horizon Dilemma

(i) CMB regions that are causally disjoint have same T to $\sim 10^{-5}$.



(ii) Structures we see today such as galaxy clusters entered horizon after decoupling epoch: $d_H(t) = r_{cluster}$.

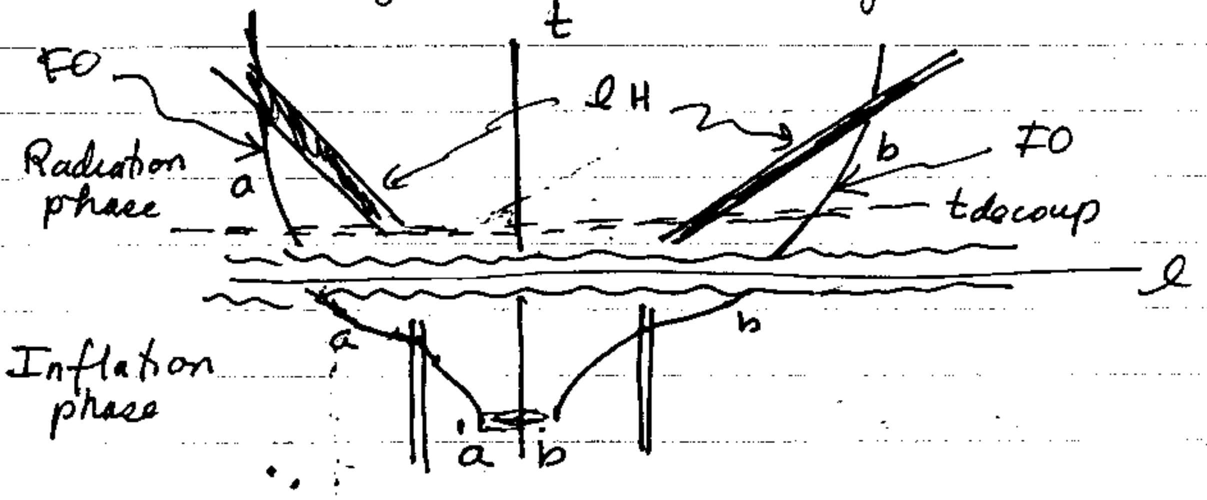


So, initial fluctuations would have no causal origin; i.e., they were <u>not</u> generated dynamically. Rather, in this picture they would be set down as some mysterious initial condition.

(A) Implication

- Points outside horizon today, or even at earlier epochs (such as CMB regions) must have been inside the horizon at earlier epochs.

- Conjecture: There must have been an earlier phase when proper distance to comoving FO increased more rapidly with time than horizon radius. In this "inflationary" phase FOs within horizon accelerate beyond it later on.



- Figure: (i) Regions a ⟺ b might be some lump of matter initially in causal contact.
  (ii) a and b accelerate outside horizon so that at decoupling they are not in horizon.
  (iii) Earlier contact would explain same CMB temperature.

- Dynamical Implications

Recall $R = \dfrac{l_{FO}(t)}{l_H(t)} = \dfrac{\dot{a}\chi}{c}$

So for $R(t)$ to increase with time, $\boxed{\ddot{a} > 0}$

Universe accelerates. Thus, if inflationary period persists for sufficiently long time, region initially within horizon can expand to be many orders of magnitude larger than initial size.

III Flatness : Weinberg argument

1^{st} Friedmann Equation:

$$H^2(t) = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

$$1 = \frac{8\pi G\rho}{3H^2(t)} - \frac{c^2 k}{a^2 H^2(t)}$$

$$1 = \Omega(t) + \Omega_K(t)$$

where $\boxed{\Omega_K(t) = -\frac{kc^2}{\dot a^2}}$

Present-day observations suggest $\Omega_K(t_0) \simeq 0$ or at least that $|\Omega_K(t_0)| \ll 1$.

- Matter domination : Since time at which $T < 10^4 K$, (t_{eq})

  $a(t) \propto t^{2/3} \Rightarrow \dot a \propto t^{-1/3}$

  $$\therefore |\Omega_K(t)| \propto t^{2/3} \propto a(t) \propto \frac{1}{T}$$

  Thus $\frac{\Omega_K(t_{eq})}{\Omega_K(t_0)} = \left(\frac{T_0}{T}\right) \Rightarrow \Omega_K(t_{eq}) < 10^{-4}$

- Radiation dominated

at $t < t_{eq}$; $a(t) \propto t^{1/2}$: $\dot{a} \propto \frac{1}{t^{1/2}}$

$$\Rightarrow |\Omega_K(t)| \propto t \propto \frac{1}{T^2}$$

Compare with $\Omega_K(T=10^{10}K)$

$$\Rightarrow \frac{\Omega_K(T=10^{10})}{\Omega_K(T=10^4)} = \left(\frac{10^4}{10^{10}}\right)^2 = 10^{-12}$$

$$\Rightarrow |\Omega_K(T=10^{10})| < 10^{-16}$$

- So curvature term would have to be even smaller, incredibly small at earlier times. Why is this true?

Guth Insight : Picture I drew before indicated that during inflation $H = \frac{\dot{a}}{a} \approx$ const. as a result:

$$|\Omega_K| = \frac{c^2}{\dot{a}^2} = \frac{c^2}{H^2 a^2} \propto \frac{1}{a^2}$$

So as universe expands dramatically $|\Omega_K|$ would drop drastically. So even if we started out with $|\Omega_K| \leq O(1)$, by the end of inflation $|\Omega_K| \longrightarrow 0$. That is if radiation phase preceded by sufficiently long period of inflation, curvature term would start out with $\Omega_K = 0$!

Quantitative: Suppose $|\Omega_K| \approx 1$ at beginning of inflation. During inflation scale factor $a(t)$ increased by factor e^N.

contained within horizon back then

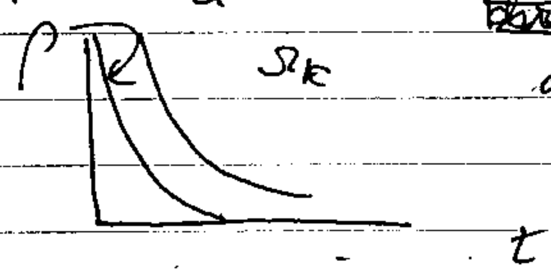
Back to General Picture

Consequences of exponential expansion

- Curvature: We saw that
  $\Omega_K \propto \frac{1}{a^2}$ so e^{Ht} expansion

  wipes out curvature ~~rapp~~ rapidly owing to large number of e-foldings. $Ht_I \gg 1$
- Density
  $\rho \propto \frac{1}{a^4} \propto \frac{1}{a^3}$. Relativistic density of ~~[illegible]~~ matter rapidly driven to zero relativistic or not

so



Origin of matter: So if $\rho_m \to 0$ at t_I, where does matter in current Universe originate?

Answer: During inflation we must have large positive energy-density vacuum ~~of~~ scalar field from which new matter is created at the end of inflation (i.e., $t = t_I$)

Homogeneous Scalar Field, ϕ

Lagrangian: $\mathcal{L} = \frac{1}{2} g^{ab} \phi_{,a} \phi_{,b} - V(\phi)$
where $V(\phi)$ is potential energy term

In case of vanishing spatial derivatives:
$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Stress energy tensor: $T_{ab} = \phi_{,a}\phi_{,b} - g_{ab}\mathcal{L} = \phi_{,a}\phi_{,b} - g_{ab}\left[\frac{\dot{\phi}^2}{2} - V(\phi)\right]$

Energy density: $\rho = T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

Pressure $P = T_{11} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$

Assume: $V(\phi)$ is slowly varying function of ϕ.
Also let V be sufficiently large that

$$\dot{\phi}^2 \ll V(\phi) \quad \text{: justify later}$$

In that case:
$$\rho \simeq V \; ; \; P \simeq -V \Rightarrow \boxed{P = -\rho}$$

Energy Conservation:

$T^{ab}{}_{;b} = 0 \Rightarrow \dot{\rho} = -\frac{3\dot{a}}{a}(\rho + P)$ and eq. of state

$$\Rightarrow \dot{\rho} = 0 \Rightarrow \boxed{\rho = \text{const}}$$

Consequences

① Field energy density = constant

Scale Factor

② Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

Since $\rho + 3P = \rho + 3(-\rho) = -2\rho$

we have $\frac{\ddot{a}}{a} = +\frac{8\pi G}{3}\rho$

Since $\rho > 0 \Rightarrow \ddot{a} > 0$: acceleration.

Let $\frac{8\pi G}{3}\rho = H^2$

$\Rightarrow \frac{\ddot{a}}{a} = H^2 \Rightarrow a(t) = \text{const}\, e^{Ht}$

③ Thus condition $\dot{\phi}^2 \ll V(\phi)$ ensures ~~expo~~ exponential expansion!

Horizon length

Recall: $l_H(t) = \frac{1}{H} = \text{const}.$

This explains why I held $l_H(t) = \text{const}$