Cosmology 2009

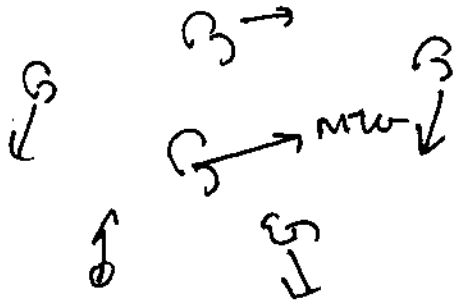
I - Introduction :

Modern cosmology began in the early 20^{th} century with Hubble's discovery that the Universe was expanding. Hubble had been determining the distances to several galaxies by recognizing in them, variable stars and other objects found in the Milky Way. Assuming these are the same objects he obtained their distances from the relationship

$$F = L/4\pi d^2$$

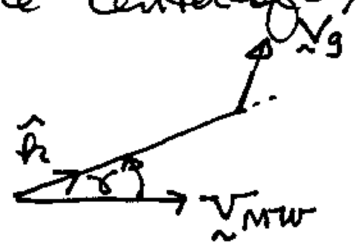
since he measured flux and L for such objects was known.

Expectation: Galaxy motions would be random like this:



Milky Way moves through a random swarm. When reflective motion of the Milky Way is removed, he expected to find small residual motions (+ and -).

Result: In the center-of-mass frame



$|\hat{r}| = 1$

Only velocity projected along the l.o.s can be measured (from Doppler shifts of spectral lines). There are two contributions:

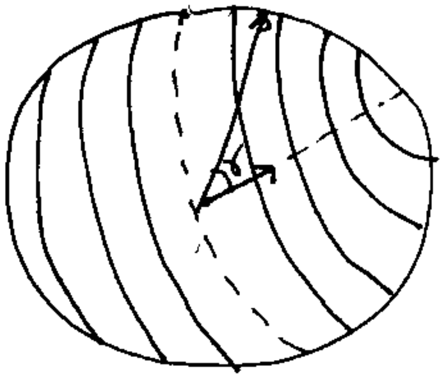
(1) peculiar motions of each galaxy, $\underset{\sim}{V}_g^i$

(2) Motion of sun through swarm

Observed velocity:

$$(V_{obs})_i = \underset{\sim}{v}_g \cdot \hat{k} - V_{MW} \cdot \hat{k} = (v_g)^i_{los} - V_{MW} \cos(\gamma)$$

To determine solar motion, divide celestial sphere into a series of rings, each with given γ (polar angle from apex of MW motion)

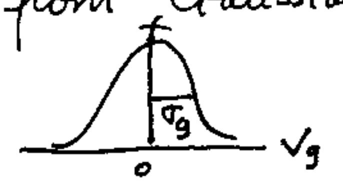


For ~~each~~ ring compute mean value of observed velocity:

$$\langle V_{obs} \rangle_\gamma = \frac{1}{N_\gamma} \sum_{i=1}^{N_\gamma} (V_{obs})_i$$

$$\langle V_{obs} \rangle_\gamma = \frac{1}{N_\gamma} \sum_{i=1}^{N_\gamma} (V_g)^i_{los} - V_{MW} \cdot \cos\gamma$$

Assume random velocities drawn from Gaussian with ϕ mean and dispersion σ_g.

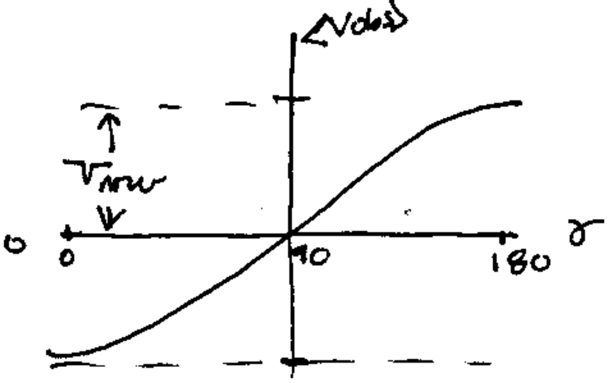


Since the rms error in the mean $\sigma_{mean} = \dfrac{\sigma_g}{\sqrt{N_\gamma}}$, we have

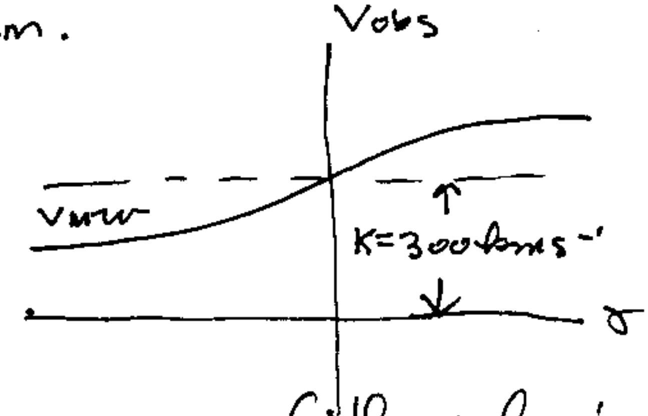
$$\langle V_{obs} \rangle_\gamma \simeq \frac{(\sigma_0)_{los}}{\sqrt{N_\gamma}} - V_{MW} \cos\gamma$$

Reasonable to assume $(\sigma_g)_{los} \sim V_{MW}$. Therefore second term dominates the first when N_g is large. Thus, the expectation was:

$$\langle V_{obs} \rangle = -V_{MW} \cdot \cos\gamma$$



Instead what he found was sinusoidal term superposed on a finite constant term.



{$V_{MW} = 280\ km\,s^{-1}$
apex: $l = 65°$, $b = 18°$}

Modern data – $V_{MW} = 308\ km\,s^{-1}$
$l = 105°$, $b = 70$

Implication: Systematic recession of the galaxies.

<u>Interplay with theory</u>:

At the time (1926–1929) the leading cosmological models based on Einstein's modified field equations

(1) Einstein Static Universe (no recession)

(2) de Sitter Universe (recession)

The new recession term favored the <u>de Sitter model</u>, but the model was non-physical in that mass density $\rho = 0$. But it made Major prediction: $K \propto d$: more distant galaxies should recede faster: $\langle V_{obs} \rangle_\gamma = K - V_{MW} \cos\gamma$

<u>Hubble</u>: To decide between the models, Hubble tried to get distances to even more distant galaxies.

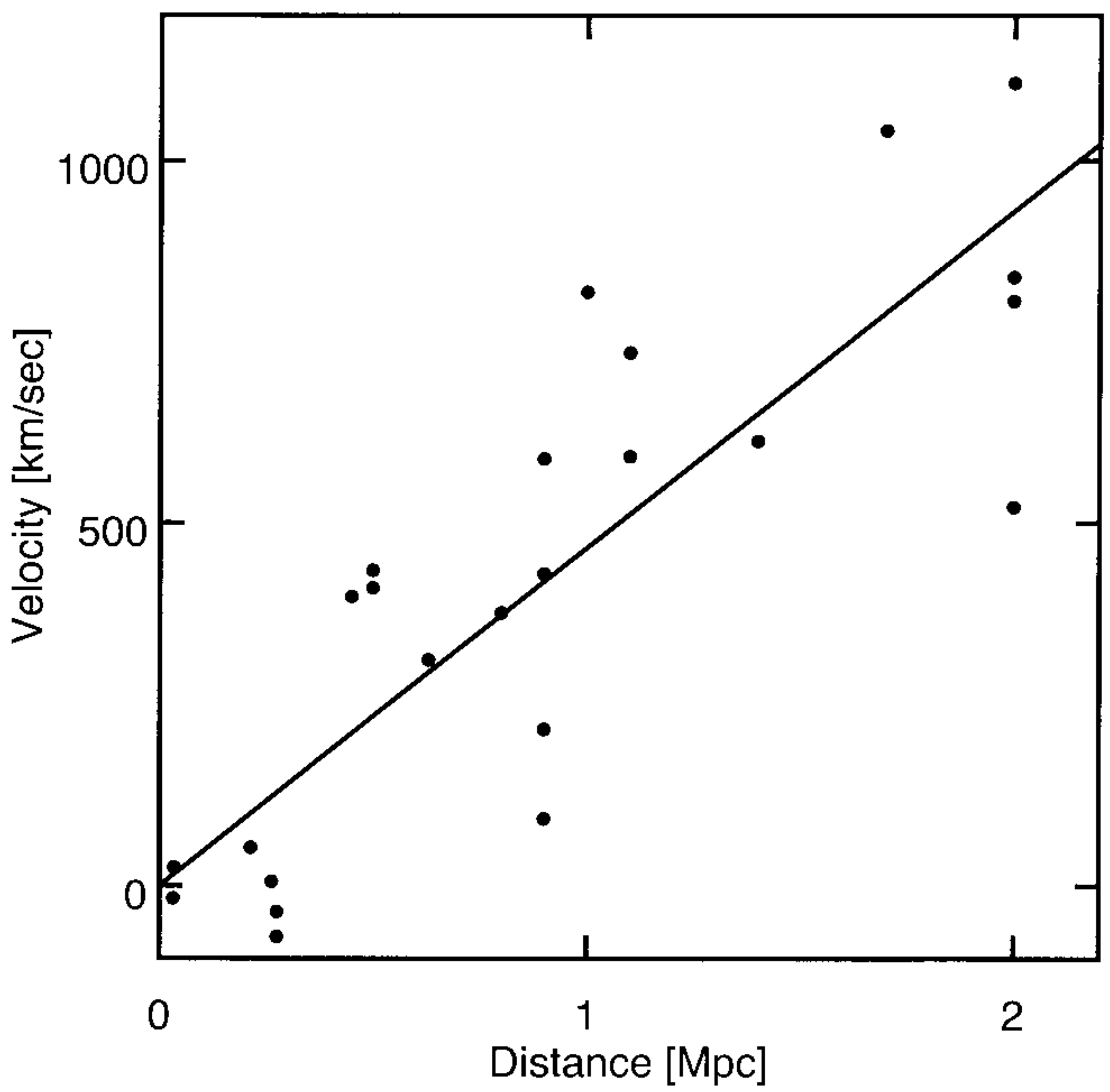


Fig. 1.— Hubble's data in 1929.

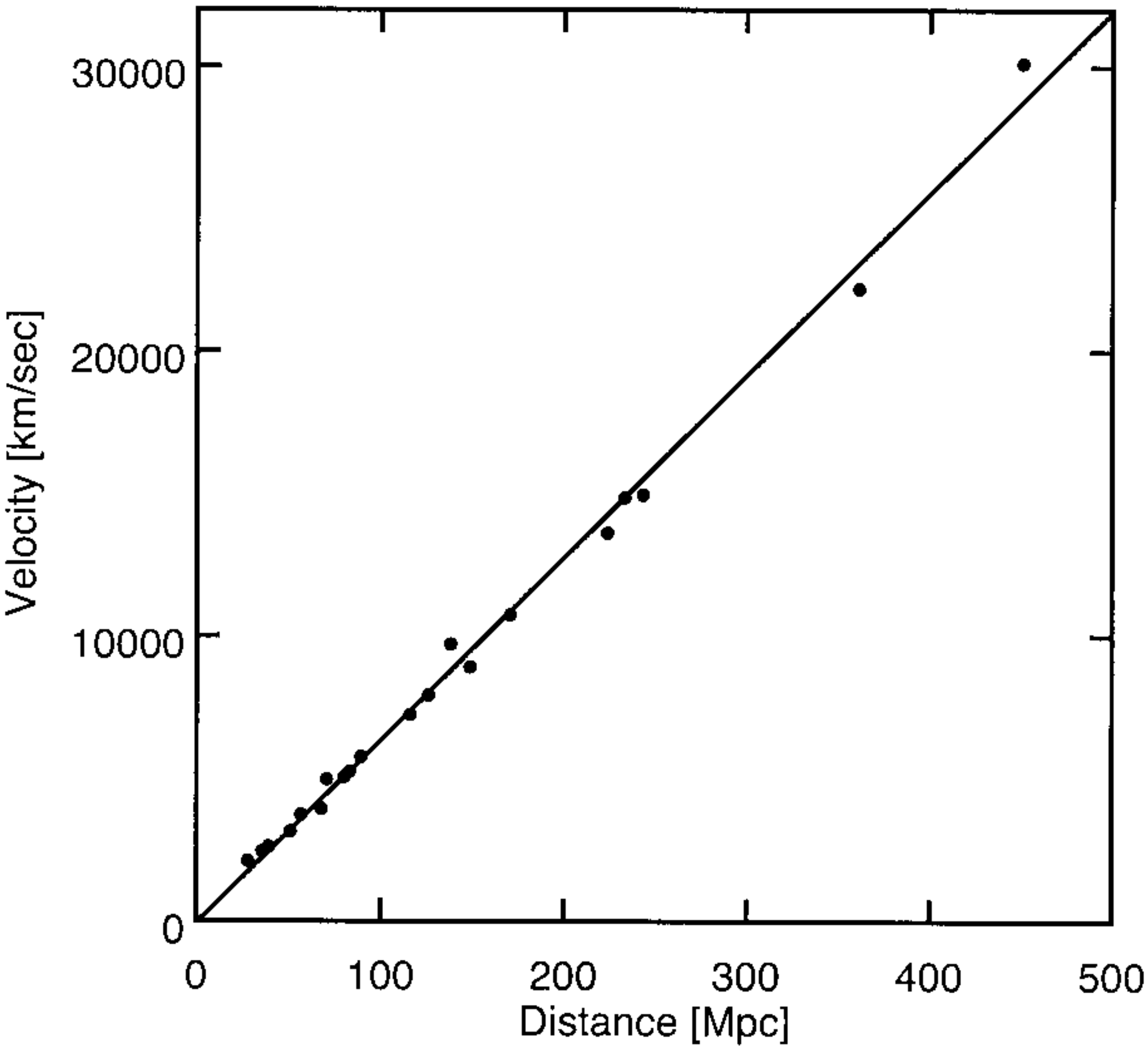
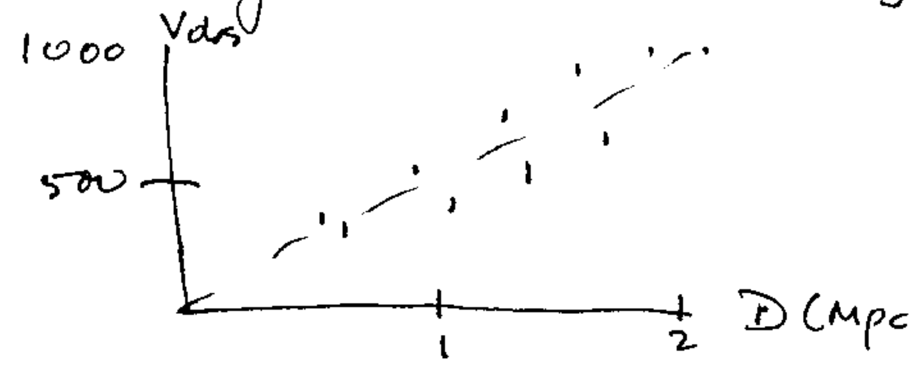


Fig. 2. Distance *vs.* redshift for Type Ia Sne

Hubble used what he thought were the brightest stars to get distances of galaxies out to the Virgo Cluster.

Result:



$$V_{obs} = K' \cdot D \quad , \quad \text{where } K' = 500 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

- Hubble's result was immediately verified
- Today we know that K' is a factor of ~ 10 smaller: K' (or H_0 as it is now known) was overestimated because Hubble confused brightest stars with more luminous HII regions, leading him to underestimate D (recall $D \propto (L/F)^{1/2}$).
- Modern Work: Using SNIa as distance indicators, one can push out much further. Predicted slope of Hubble law $V_{obs} = H_0 \cdot D$ verified out $D = 500$ Mpc!

$$H_0 = 72 \pm 8 \text{ km s}^{-1} \text{ Mpc}^{-1} \text{ (Key Project)}$$
$$h = H_0 / 100 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

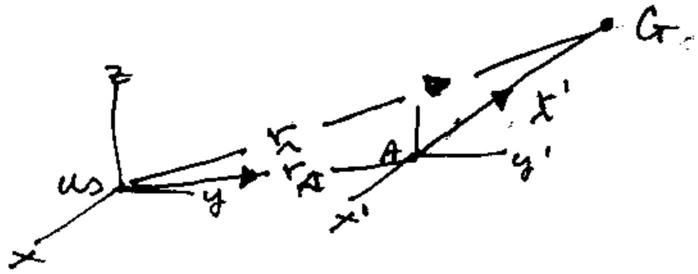
Implications:

1. Universe is expanding.
2. Linear dependence on distance is special. Let's generalize form to

$$\underset{\sim}{v} = H_0 \underset{\sim}{r}$$

Rotation (ii) Hubble's law independent of direction since $|\underset{\sim}{v}|$ depends on $|\underset{\sim}{r}|$ which is invariant under rotation group.

Translation (iii) Form of Hubble's law also invariant under translation:



We observe galaxy G at $\underset{\sim}{r}$

Astronomer A observes galaxy G at r'

$$\underset{\sim}{r}' = \underset{\sim}{r} - r_A$$

$$\underset{\sim}{v}'(\underset{\sim}{r}') = v(\underset{\sim}{r}) - v(\underset{\sim}{r}_A)$$

But since $\underset{\sim}{v}(r) = H_0 \underset{\sim}{r}$; $\underset{\sim}{v}(r_A) = H_0 r_A$

$$\underset{\sim}{v}'(r') = H_0 \underset{\sim}{r} - H_0 r_A$$

$$= H_0 (\underset{\sim}{r} - \underset{\sim}{r}_A)$$

$$\therefore \boxed{\underset{\sim}{v}'(\underset{\sim}{r}') = H_0 \underset{\sim}{r}'}$$

A sees same Hubble as we do, but measures different recession velocity for galaxy G.

(iii) All observers everywhere see Universe expand around them: they all think they are at the center.

Implication: there is no preferred center. The Universe is homogeneous as well as isotropic

Pause to reflect:

what do we mean when we ~~[illegible]~~ make statements like:

(1) The universe is expanding?

or (2) the universe is homogeneous & isotropic?

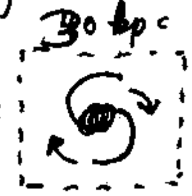
Universal facts: To answer such questions or, stated differently, to test such hypotheses we need to apply them to a "fair sample" of the universe to determine whether these properties are truly universal or just local properties.

Examples

(I) **Kinematics**

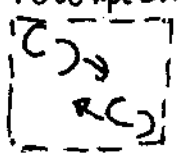
What is state of motion of the Universe? The answer clearly depends on the size of the region over which we average the velocity fields:

galaxy 30 kpc



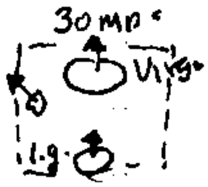
On 30 kpc, we are looking at self-gravitating galaxy where dominant motion is rotation

Local Group



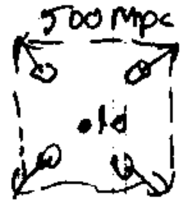
On 1 Mpc scale, motions dominated by infall of M31 and Galaxy: Local Group.

Virgo supercluster



Expansion + gravitational in-fall of local group toward Virgo Cluster { Hubble's original sample

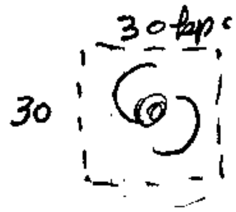
Fair Sample



It is only when we ~~average~~ average out motions on scales ~30 Mpc or larger & examine ~ 500 Mpc expansion will dominate

(II) Density

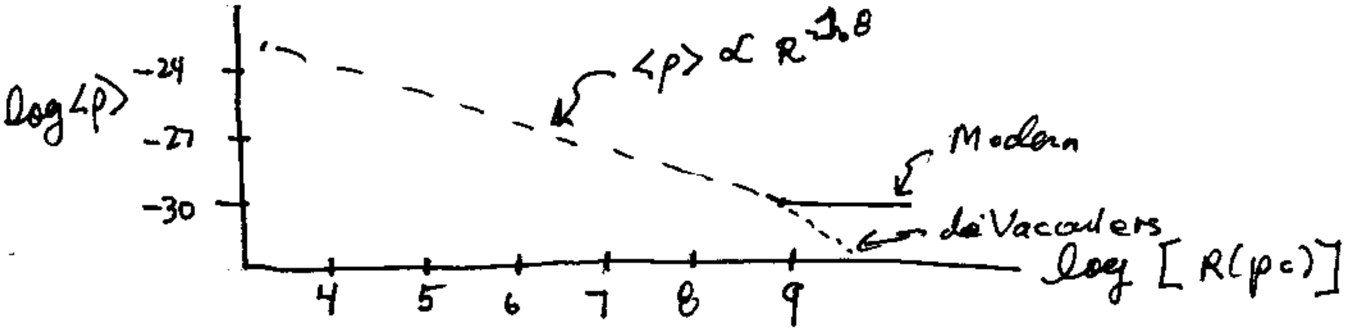
what is the mass density of the Universe?
Again to get suitable assessment one must take average over "fair sample" of the Universe.



$M(\text{galaxy}) \simeq 10^{12} M_\odot$ on radii ~~scales~~ of $\sim 15\text{-}20$ kpc

$\langle \rho \rangle = M(\text{galaxy}) / 4\pi R^3/3 \simeq \text{few} \times 10^{-24}\ \text{g cm}^{-3}$

But we know $\langle \rho \rangle$ is <u>not</u> universal since it decreases with increase in sample size!



Minimum R required to establish a universal mean density is about same as required to establish cosmic expansion, ~ 300-500 Mpc!

Back to Expansion: GR Models

It is important to emphasize that expansion of the universe was predicted in 1922 by the Russian mathematician, Alexander Friedmann. He solved the Einstein Field Equations for

(a) Riemann 3D space with constant curvature, i.e., a space that is homogeneous and isotropic.

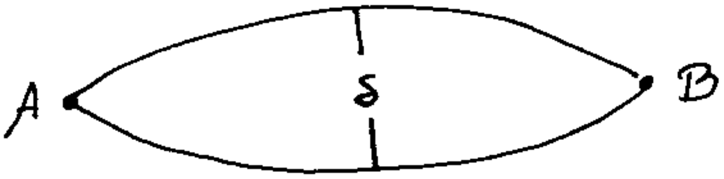
(b) an 4D spacetime that is expanding

First, ~~let's~~ examine geometry of 3-D space and then we'll examine 4D spacetime.

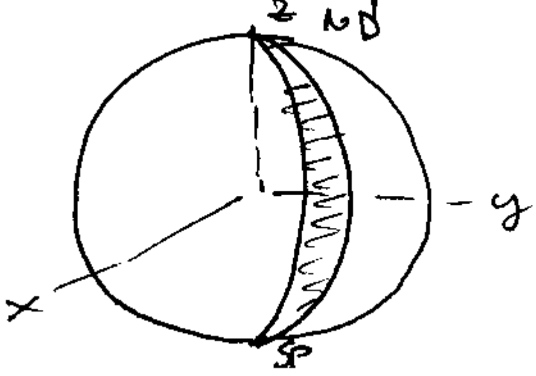
- Metric: Space admits a metric $g_{\alpha\beta}$ such that the line element or invariant interval is given by $dl^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ (summation convention α, $\alpha = 1,2,3$).
  Note: dl is ~~[illegible]~~ metric distance measured instantaneously between two points.
  Geodesics: Extremal curves between 2 points in space (or more generally spacetime)

$$\delta \int g_{ij} \frac{dx^i}{dv}\frac{dx^j}{dv}\, dv = 0$$
(More later)

~~[illegible]~~ Geodesic deviation in Space of constant (positive) curvature



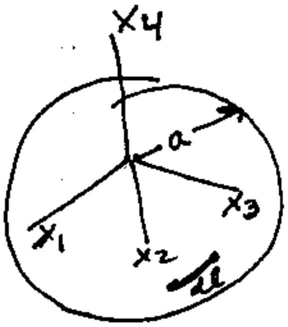
2D But this is the same property followed by the surface of 2-D sphere. Note, we can only visualize them by embedding 2-D sphere in 3-D Euclidean Space



two longitudes going from NP to SP

3D

(i) Einstein Space : 3D analogue of 2D sphere.
We can interpret this by embedding it in fictitious 4-D Euclidean Space:



3D-sphere with radius a. Therefore

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2 \quad (1)$$

Euclidean line element

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \quad (2)$$

On 3D surface, a = const. Therefore derivative of eq. (1) equals zero:

$$x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3 + x_4\,dx_4 = 0$$

Solve for dx_4!

$$x_4\,dx_4 = -(x_1dx_1 + x_2dx_2 + x_3dx_3)$$

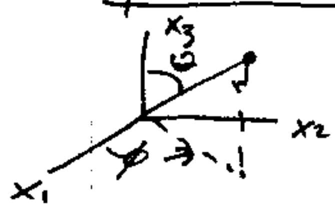
$$dx_4^2 = \frac{(x_1dx_1 + x_2dx_2 + x_3dx_3)^2}{x_4^2}$$

Eliminate x_4 : $$dx_4^2 = \frac{(x_1dx_1 + x_2dx_2 + x_3dx_3)^2}{a^2 - x_1^2 - x_2^2 - x_3^2}$$

Therefore dl^2 on surface of 3D sphere is:

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1dx_1 + x_2dx_2 + x_3dx_3)^2}{a^2 - x_1^2 - x_2^2 - x_3^2}$$

(ii) Spherical Coordinates



$$r^2 = x_1^2 + x_2^2 + x_3^2$$

$$r\,dr = x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3$$

and : $$dx_1^2 + dx_2^2 + dx_3^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

Therefore: $$dl^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2 + \frac{(r\,dr)^2}{a^2 - r^2}$$

As a result:

$$d\ell^2 = dr^2\left[1 + \frac{r^2}{a^2-r^2}\right] + r^2 d\Omega : \quad d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$= dr^2\left[\frac{a^2}{a^2-r^2}\right] + r^2 d\Omega$$

$$= \frac{dr^2}{1-\left(\frac{r}{a}\right)^2} + r^2 d\Omega$$

or $$d\ell^2 = a^2\left[\frac{d\left(\frac{r}{a}\right)^2}{1-\left(\frac{r}{a}\right)^2} + \left(\frac{r}{a}\right)^2 d\Omega\right]$$

or in terms of $r' \to r/a$ rescaled coordinates

$$\boxed{d\ell^2 = a^2\left[\frac{dr^2}{1-r^2} + r^2 d\Omega\right]} \quad (3)$$

More generally:

$$d\ell^2 = a^2\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right]$$

$$K = \begin{cases} +1 & : \text{Positive curvature} \\ 0 & : \text{flat} \\ -1 & : \text{negative curvature} \end{cases}$$

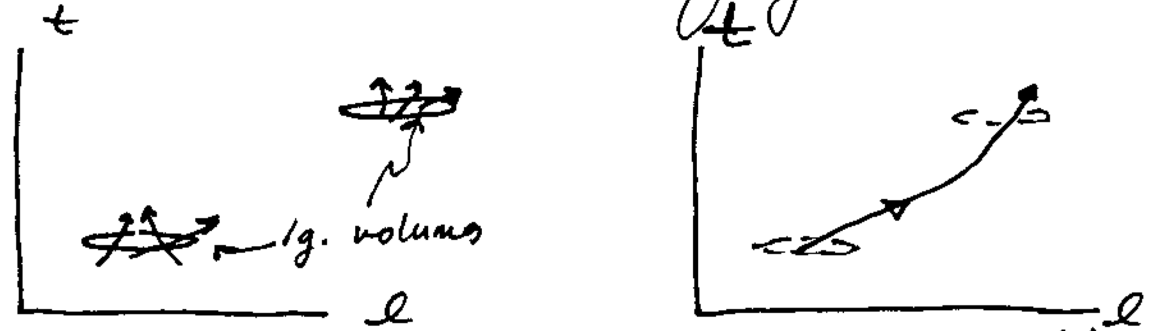
One of the principal goals of cosmology has been to determine **K**. We shall discuss this a great deal in future lectures.

Before I go on to interesting properties of these spaces of constant curvature, let's discuss the spacetime metric

Kinematics of Homogeneous Perfect fluids

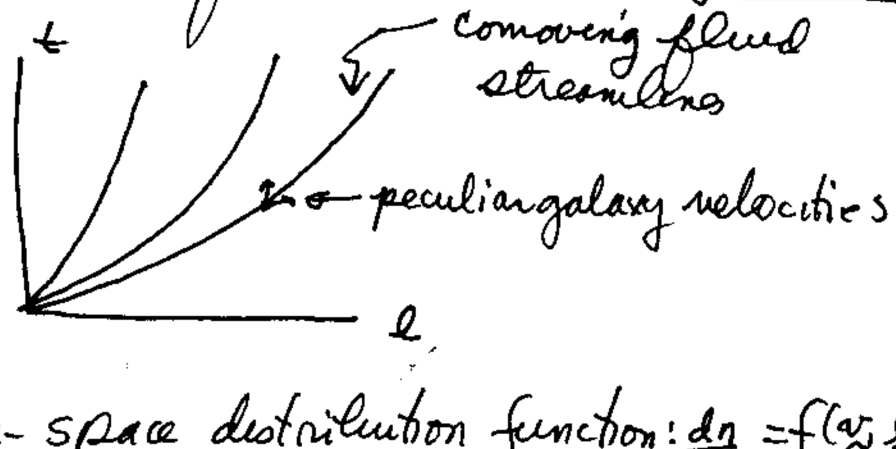
We've already discussed concept of fair sample. What this means is that we represent motion of the universe by time-dependent flow of "fluid" streamlines. Thus motion at a point is phase-space average over actual motions of galaxies.

Spacetime Diagram



Actual galaxy worldlines — fluid representation

Averaging over large volumes, irregularities disappear. And there is a unique mean velocity at each (l,t) point



Define phase-space distribution function: $\frac{dn}{n} = f(\underset{\sim}{v}, \underset{\sim}{l})\, d^3\underset{\sim}{v}\, d^3\underset{\sim}{l}$

Average fluid velocity:

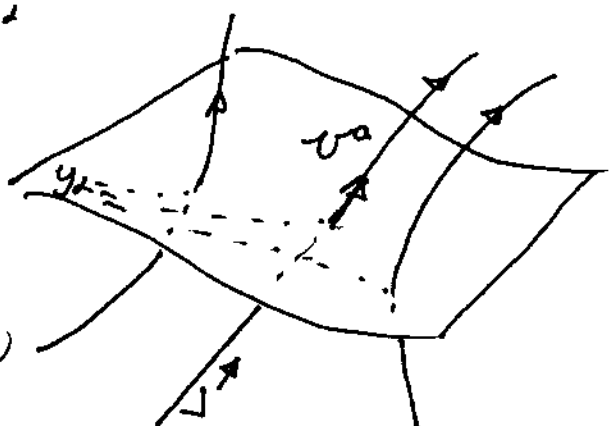
$$u = \bar{v}(\ell, t) = \frac{\iint_R v\, f(\underset{\sim}{v}, \underset{\sim}{\ell})\, d^3v\, d^3\ell}{\iint_R f(\underset{\sim}{v}, \underset{\sim}{\ell})\, d^3v\, d^3\ell} \qquad (4)$$

Curve Congruences that are hypersurface orthogonal

(1) Given a unique velocity vector field v^a where u^a is 4-velocity ($a = 0, 1, 2, 3$)
↑ time ↑ space

- Draw curves to which u^a is tangent
- Since u^a is unique, only 1 curve goes through each spacetime point

(2) Consider 3D hypersurface intersected once by each curve: 3D spheres are such hypersurfaces. If y^α are 3D coordinates within each hypersurface, we can label each curve by its intersection coordinate y^α



(a) Let y^α = const along each curve. y^α are thus comoving coordinates

(b) If $ds^2 = g_{ab}dx^a dx^b$ $\quad a = 0, 1, 2, 3$ ~~thus tangent~~ and v is parameter along curves, Recap: I finished previous lecture by introducing two concepts:

(1) 3D spaces of constant curvature - We discussed their properties by embedding 3D sphere in a fictitious 4D Euclidean space:

3D line interval: $dl^2 = a^2 \left[\frac{dr^2}{1-r^2} + r^2 d\Omega \right]$

$d\Omega = d\theta^2 + \sin^2\theta \, d\phi^2$

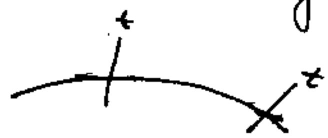
More generally: $dl^2 = a^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega \right]$ $\left(k = \begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix} \right)$

(2) 4D Spacetime - The 3D spaces from (1) have properties indicating no

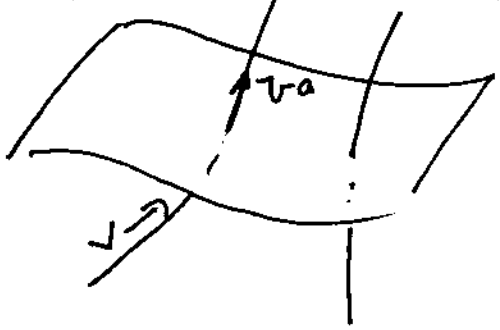
(a) preferred center

(b) preferred direction

Thus there should be global time that is synchronous for all points in 3D space (hypersurface). But unlike SR, there is in general no uniform time in GR



(3) Hypersurface Orthogonal flows - We sliced up spacetime with a sequence of spacelike hypersurfaces (3D volumes) each of which is parameterized by some function, which, as we shall see, turns out to be cosmic time.



Let's consider flow lines to represent the timelike trajectories of fundamental observers. In that case we let $v = \tau$ where τ is proper time. Proper time is a Lorentz invariant parameter that measures "distance" along world line measured by clocks carried by Fundamental Observer. We define 4 velocity $v^a \equiv dx^a/d\tau$ which are tangent vectors to F.O. world lines fr

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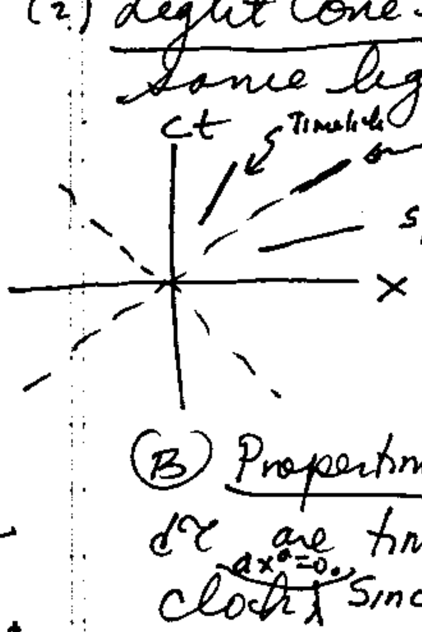
Aside:

(1) Local Inertial Frame: In GR we can always transform $g_{ab}(x^c)$ into Minkowski metric at a given spacetime event. Specifically $g_{ab} \to \eta_{ab}$ in a local patch (Same as principle of equivalence: cannot distinguish free fall from constant velocity motion by local experiments).

In SR: $ds^2 = \eta_{ab}dx^a dx^b = c^2dt^2 - dx^2 - dy^2 - dz^2$

(2) Light Cone Structure: Therefore GR inherits the same lightcone structure as SR.



(A) along x axis $ds^2 = c^2dt^2 - dx^2$

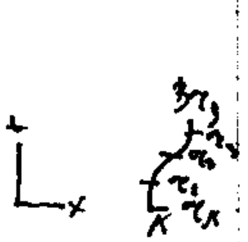
a) Timelike: $c^2dt^2 > dx^2 \Rightarrow ds^2 > 0$

b) lightlike: $c^2dt^2 = dx^2 \Rightarrow ds^2 = 0$

c) spacelike: $c^2dt^2 < dx^2 \Rightarrow ds^2 < 0$

(B) Propertime: Define $d\tau$ by $ds^2 = c^2 d\tau^2$

$d\tau$ are time intervals measured by observer carrying clock ($dx^a = 0$). Since ds^2 is Lorentz invariant so is $d\tau$



$\therefore\ c^2 d\tau^2 = g_{ab}\, dx^a dx^b > 0$: Timelike

$c^2 = g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = g_{ab}v^b v^a = v_a v^a$ or $g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 1$

Then if $v = s$, tangent vectors to curves are time-like velocities $u^a = \frac{dx^a}{ds}$

since $g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 1 \Rightarrow g_{ab}v^a v^b = 1$ or $v_a v^a = 1$ ← 4 scalar product

Surface-forming Congruences

Under what circumstances are there hyper-surfaces to which v_a is orthogonal?

(1) Hypersurface equation $f(x_a) = \text{const.}$ $\Big\{$ Defines 3D surface [redacted] in 4D spacetime

(2) For any [redacted] displacement $d\hat{x}^a$ within hypersurface

$f(x^a + d\hat{x}^a) = f(x^a) = \text{const}$ But this implies

$f(x_a) + \frac{\partial f}{\partial x^a} d\hat{x}^a = f(x^a) \Rightarrow \boxed{\frac{\partial f}{\partial x^a} d\hat{x}^a = 0}$ (5)

$\frac{\partial f}{\partial x_a} \perp$ to surface S.

(3) As a result: $v_a \propto f_{,a}$. Or v_a must be curl free ([redacted] no vortices allowed)

(4) For [redacted] arbitrary displacement dx^a (not on S)

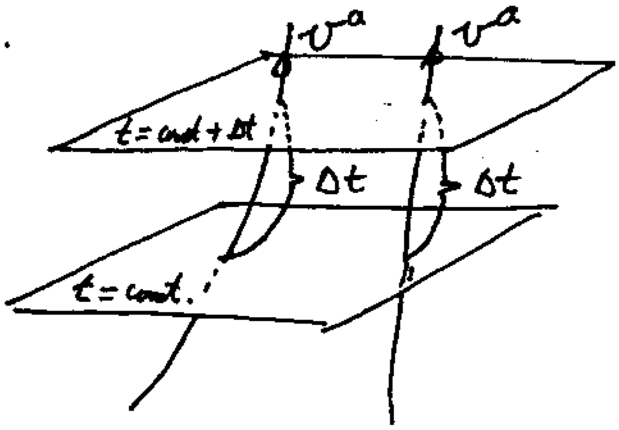
$$v_a dx^a = g(x^a)\, df$$

(5) If f is a spacelike hypersurface, [redacted] then v_a points in timelike direction. To preserve homogeneity set $g(x^a) = \text{const} = 1$.

therefore $v_a dx^a = df \equiv dt$

dt is cosmic time interval.

Cosmic Time:



Cosmic time chosen such that all clocks are synchronized on a given hypersurface. Cosmic time is defined.

Spacetime Metric

$$ds^2 = g_{ab}(x^c)\, dx^a dx^b$$

Define projection operator $h^{ab} = v^a v^b - g^{ab}$

therefore: $v_a h^{ab} = v_a v^a v^b - v_a g^{ab} = v^b - v^b = 0$

Thus tensor h^{ab} projects vectors $\perp$ to v^a

therefore: $ds^2 = (v_a v_b - h_{ab}) dx^a dx^b = (v_a dx^a)^2 - \underbrace{h_{\alpha\beta} dx^\alpha dx^\beta}_{\text{only in hypersurfaces}}$

$\alpha = 1, 2, 3$

$a = 0, 1, 2, 3$

Choose homogeneous flows that are hypersurface orthogonal.

$$v_a dx^a = dt$$

$$\boxed{\text{Therefore: } ds^2 = dt^2 - h_{\alpha\beta} dx^\alpha dx^\beta} \quad (6)$$

10k or $ds^2 = dt^2 - dl^2$ {for const co-moving} when $dl = 0$

dt is proper time = global time since $ds = c\,d\tau = dt$

We wind up with Robertson-Walker Metric

$$\boxed{ds^2 = dt^2 - a^2_{(t)}\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right]} \quad (7)$$

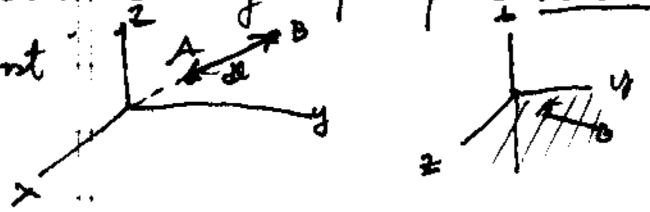
Important to remember that r, θ, ϕ are comoving coordinates: they are labels that don't change in time. Remember also they apply to special class of "fluid" observers; i.e, fundamental observers. These are the special class of observers (i.e., coordinate frames) that see a perfectly isotropic Universe around them.

Let's examine some consequences of eq. (7)

(A) Hubble Law:

Consider two fundamental observers (FOs) separated by proper radial distance dl at time t

$t = \text{const}$



i.e., dl is an instantaneous ruler.

Because $d\theta = d\phi = 0$ and $dt = 0$. From eq. (7) $ds^2 = dt^2 - dl^2$ we have

$$dl = \frac{a(t) \cdot dr}{\sqrt{1 - kr^2}}$$

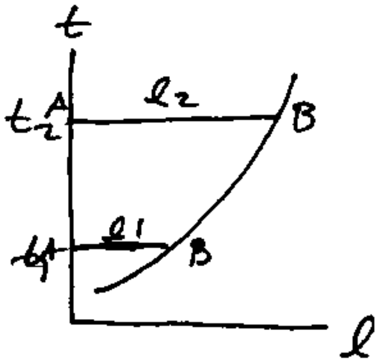
If $k = 1$, assume $r \ll 1$ ($\Leftrightarrow r' \ll a$): distances small compared to radius of curvature.

In that case $dl \simeq a(t)\, dr$

Let FO A be at $r=0$ and FO B be at r
Integrate at $t=$ const.

$$\int_0^{l} dl = a(t)\int_0^{r} dr' \Rightarrow \boxed{l(t) = a(t)\, r} \quad (8)$$

Spacetime Picture



Since $a(t)$ increases with time due to cosmic expansion. But coordinate label $r=$ const.
Spacetime diagram of separation between A & B is shown in spacetime picture

Velocity: Recessional velocity of FO is given by

$$u = \frac{dl}{dt} = \frac{d}{dt}[a(t) r]$$

$$u = \dot{a} \cdot r$$

But since $r = l(t)/a(t)$, we have

$$u(t) = \frac{\dot{a}}{a}\, l(t)$$

Letting $H(t) \equiv \frac{\dot{a}}{a}$ be the Hubble parameter,

$$u(t) = H(t)\, l$$

At present $H(t_0) \equiv H_0$ which yields

$$\boxed{u = H_0 l} \quad (9) \quad \text{— Hubble's law}$$

Natural byproduct of FRW metric!

(B) Redshift

Consider a radially propagating light ray. In that case:

(1) the interval connecting 2 spacetime points along a lightlike geodesic $ds = 0$

(2) because light ray moves in radial direction, $d\theta = d\phi = 0$.

From eq. (7) we therefore have

$$dt^2 = \frac{a^2 dr^2}{1-kr^2}$$

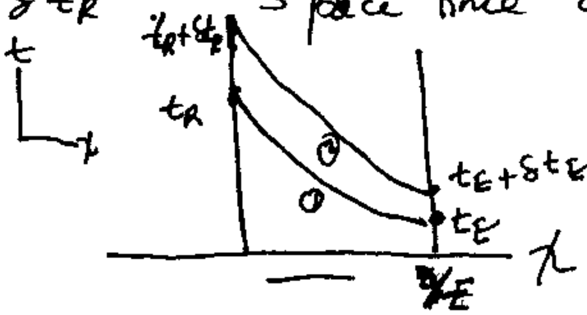
For an incoming light ray we take the negative root, so

$$dt = -\frac{a\,dr}{\sqrt{1-kr^2}} \qquad \left\{ \text{increasing time } dr \text{ decreases} \right\}$$

For convenience let $d\chi \equiv +\frac{dr}{\sqrt{1-kr^2}}$: $\chi = \int \frac{dr}{\sqrt{1-kr^2}}$

$$\boxed{\frac{dt}{a(t)} = -d\chi} \qquad (10)$$

Now consider an emitter at comoving coordinate χ_E that emits successive light pulses separated by δt_E; i.e., at t_E, $t_E + \delta t_E$. Receptor at the origin receives light pulses at t_R, $t_R + \delta t_R$; space time diagram:



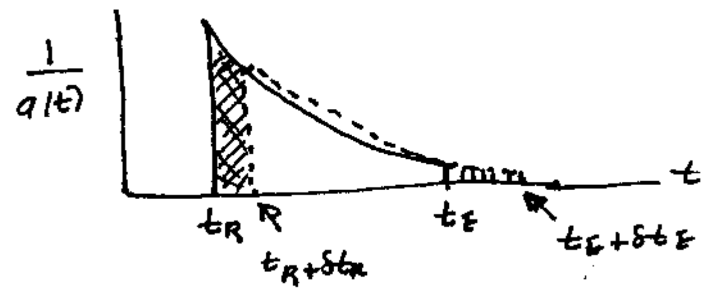
Integrate along each lightlike geodesic:

(1) $$\int_{t_E}^{t_R} \frac{dt}{a(t)} = -\int_{r_E}^{0} dr = r_E$$

(2) $$\int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{dt}{a(t)} = r_E$$

Implication: $$\int_{t_E}^{t_R} \frac{dt}{a(t)} = \int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{dt}{a(t)}$$

<u>Integrand</u>



Areas under solid integrand = area under dashed integrand only if amount subtracted ✻ = amount added ″ ; i.e., if

$$\int_{t_R}^{t_R+\delta t_R} \frac{dt}{a(t)} = \int_{t_E}^{t_E+\delta t_E} \frac{dt}{a(t)}$$

In the limit $\delta t_E \to 0$; $\delta t_R \to 0$ we have

$$\frac{\delta t_E}{a(t_E)} = \frac{\delta t_R}{a(t_R)}$$

Let the pulses be successive wave crests separated by 1 period.

of an electromagnetic wave. In that case received wavelength $\lambda_R = c \cdot \delta t_R$, and emitted wavelength. $\lambda_E = c\,\delta t_E$. As a result.

$$\frac{\lambda_R}{\lambda_E} = \frac{c\,\delta t_R}{c\,\delta t_E} = \frac{a(t_R)}{a(t_E)}$$

Since $a(t_R) > a(t_E)$, the received wavelength is redshifted. We define redshift z as

$$\boxed{1+z \equiv \frac{\lambda_R}{\lambda_E} = \frac{a(t_R)}{a(t_E)}} \quad (11)$$

Prediction: the further back in time we look, the larger the redshift. To test this hypothesis we will need definition of distance and techniques for measuring distances that are independent of redshift. We need these to test above eq. and of course Hubble's law which is predicted in limit of small $z \ll 1$.

Note: $$1+z = \frac{\nu_E}{\nu_R} \simeq \frac{\nu_R + \Delta\nu}{\nu_R} \simeq 1 + \frac{V}{C}$$

$$1+z = \frac{a(t_R)}{a(t_R - \Delta t)} \simeq \frac{a(t_R)}{a(t_R) - \Delta t(\dot{a})_R}$$

$$\therefore\ 1 + \frac{V}{c} \simeq \frac{1}{1 - \Delta t\left(\frac{\dot{a}}{a}\right)_R} \simeq 1 + \Delta t\left(\frac{\dot{a}}{a}\right)_R$$

But since $\left(\frac{\dot a}{a}\right)_R = H_0$; $\Delta t = l/c$

we have $\frac{V}{c} \simeq \frac{l}{c} H_0$ or $\boxed{V = H_0 l}$ -

Geodesics admitted by FRW metric

Let v be an "affine" parameter that varies along paths between two fixed spacetime points.

Geodesics are stationary solutions to variational principle:

$$\delta \int_A^B \underbrace{\left(g_{ab}\frac{dx^a}{dv}\frac{dx^b}{dv}\right)} dv = 0$$

Think of $g_{ab}(x^c)\frac{dx^a}{dv}\frac{dx^b}{dv} = \mathcal{L}\left(x^a, \frac{dx^a}{dv}\right)$ as a Lagrangian satisfying Lagrange's equations.

$$\boxed{\frac{d}{dv}\left(\frac{\partial \mathcal{L}}{\partial(dx^a/dv)}\right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0 \quad (12)}$$

solutions $x^a(v)$ are geodesics:

Consider radial geodesics: In that case

$$ds^2 = dt^2 - a^2(t)\, d\chi^2$$

$$\therefore \quad \mathcal{L}\left(t, \chi; \frac{dt}{dv}, \frac{d\chi}{dv}\right) = \left(\frac{dt}{dv}\right)^2 - a^2(t)\left(\frac{d\chi}{dv}\right)^2$$

Integrate and we have:

$$\frac{1}{2}\left(\frac{dt}{dv}\right)^2 = \frac{A^2}{2a^2} + B$$

or

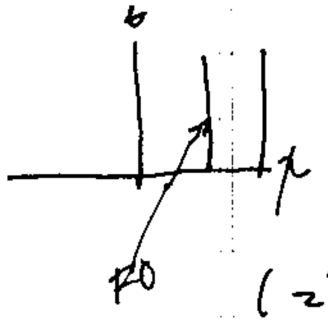
$$\boxed{\left(\frac{dt}{dv}\right)^2 = \frac{A^2}{a^2} + C} \quad (14)$$

<u>Time like Geodesics</u> : replace $v = s$ (proper time)

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 1 \quad \Rightarrow \left(\frac{dt}{ds}\right)^2 - a^2(t)\left(\frac{d\chi}{ds}\right)^2 = 1$$

$$\frac{A^2}{a^2} + C - a^2\left(\frac{A}{a^2}\right)^2 = 1 \quad \Rightarrow \quad C = 1$$

So in this case $\boxed{\frac{d\chi}{ds} = \frac{A}{a^2}\ ; \quad \frac{dt}{ds} = \pm\sqrt{\frac{A^2}{a^2} + 1}}$ (15)



(1) $A = 0$: Special class of fundamental observers for whom $\chi = $ const. are comoving ~~observers~~ coordinates. They see homogeneous & isotropic Universe. Comoving FOs moving along special class of geodesics.

(2) $A \neq 0$:

Let 3 momentum: $P = m\sqrt{U^\alpha U_\alpha}$

$i = 1,2,3$

$$P = m\sqrt{h_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}} = m\sqrt{a^2\left(\frac{d\chi}{ds}\right)^2}$$

$h_{\alpha\beta} = (\quad)$

$$P(t) = m\sqrt{a^2\,\frac{A^2}{a^4}} \propto \frac{1}{a}$$

aoq9-2

$\chi]$ $$\frac{d}{dv}\left(\frac{\partial \mathcal{L}}{\partial(d\chi/dv)}\right) - \frac{\partial \mathcal{L}}{\partial \chi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(d\chi/dv)} = -a^2 \times 2 \times \left(\frac{d\chi}{dv}\right) ; \quad \frac{\partial \mathcal{L}}{\partial \chi} = 0$$

$$\frac{d}{dV}\left[-2a^2\frac{d\chi}{dv}\right] = 0 \quad \Rightarrow \boxed{\frac{d\chi}{dV} = \frac{A}{a^2}} \therefore A = \text{const.} \quad (13)$$

$t]$ $$\frac{d}{dv}\left(\frac{\partial \mathcal{L}}{\partial(dt/dv)}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(dt/dv)} = 2\frac{dt}{dv} ; \quad \frac{\partial \mathcal{L}}{\partial t} = -2a\left(\frac{da}{dt}\right)\left(\frac{d\chi}{dv}\right)^2$$

$$\therefore \quad \frac{d}{dv}\left(\cancel{2}\frac{dt}{dv}\right) - \left(-\cancel{2}a\dot{a}\right)\left(\frac{d\chi}{dv}\right)^2 = 0$$

$$\frac{d^2t}{dv^2} = -a\,\dot{a}\left(\frac{A^2}{a^4}\right)$$ ← from eq. (13)

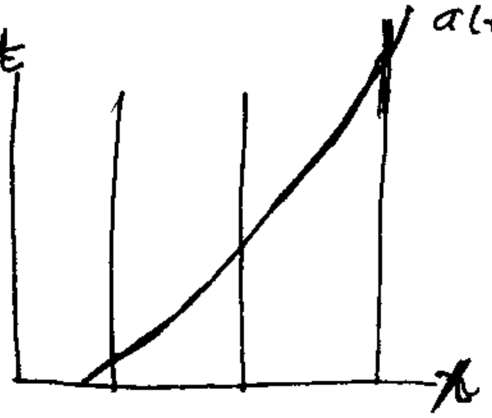
Let $$\frac{da}{dt} = \frac{da}{dV} \times \frac{1}{\frac{dt}{dV}} \Rightarrow \frac{d^2t}{dv^2} = -a\frac{\frac{da}{dv}}{\frac{dt}{dV}}\left(\frac{A^2}{a^4}\right)$$

Therefore $$\frac{dt}{dV}\frac{d^2t}{dV^2} = -\frac{A^2}{a^3}\frac{da}{dV} \Rightarrow$$

$$\frac{dt}{dV}\,d\left(\frac{dt}{dV}\right) = -\frac{A^2}{a^3}da$$

Cos99-23

Magnitude of the 4 velocity $|u| = (u^a u_a)^{1/2}$ dies off like $v(t) \propto \frac{1}{a(t)}$ as universe expands



trajectory slopes toward t axis as time goes by

Are current peculiar velocities $v \simeq 300$ km/s relicts of the past?

$$\text{Since} \quad v(t) \propto \frac{1}{a(t)} \Rightarrow \frac{v(t_{initial})}{v(t_0)} = \frac{a(t_0)}{a(t_{initial})}$$

We shall see that $1 + z_{initial} \simeq 10^3$ is an important epoch from which CMB radiation was emitted. If current peculiar motions are relicts of the past, $v(\text{initial}) \to c$, these would generate large Doppler shifts in emitted radiation, which would in turn give rise to enormous intensity or Temp fluctuations in CMB, which are not observed.

Better Explanation : Current peculiar velocities due to local accelerations caused by gravitational fields generated by irregularities in mass distribution.