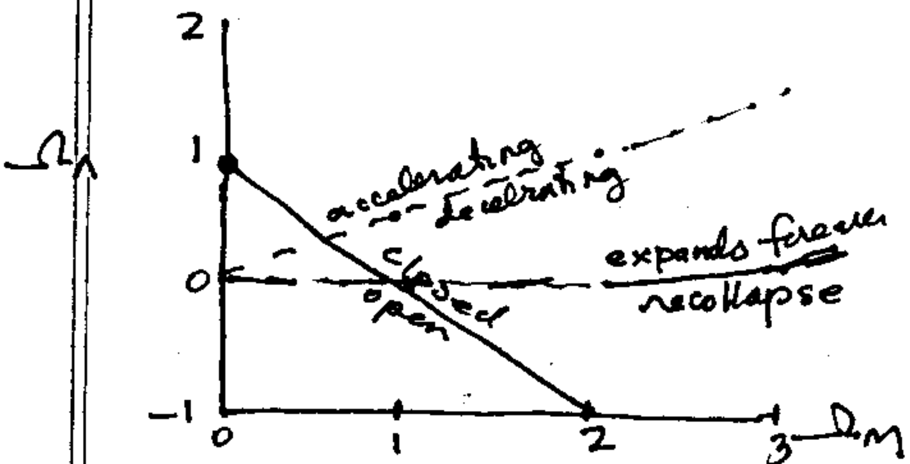
Add comments about previous week's lecture

(1) Ω_Λ vs Ω_m diagram



• $\Omega_\Lambda = 0$
$\Omega_m > 1$: recollapse: K
$\Omega_m \le 1$: expand as forever

• $\Omega_\Lambda \neq 0$: changes everything

We discussed two boundary lines

(A) $\Omega_\Lambda = 1 - \Omega_m$: closed versus open

(B) $\Omega_\Lambda = \frac{1}{2}\Omega_m$: decelerating versus accelerating

But $\exists$ a third line between models that expand forever and models that expand to a max and then recollapse.

Straightforward way to look for this is to determine whether $\dot{a} = 0$ at finite time or finite $x = a(t)/a_0$

Recall: $$\frac{1}{H_0}\left(\frac{1}{a}\frac{da}{dt}\right) = \frac{H(z)}{H_0} = \sqrt{\Omega_\Lambda + \frac{\Omega_m}{x^3} + \frac{\Omega_R}{x^4} + \frac{(1-\Omega_m-\Omega_R-\Omega_\Lambda)}{x^2}}$$

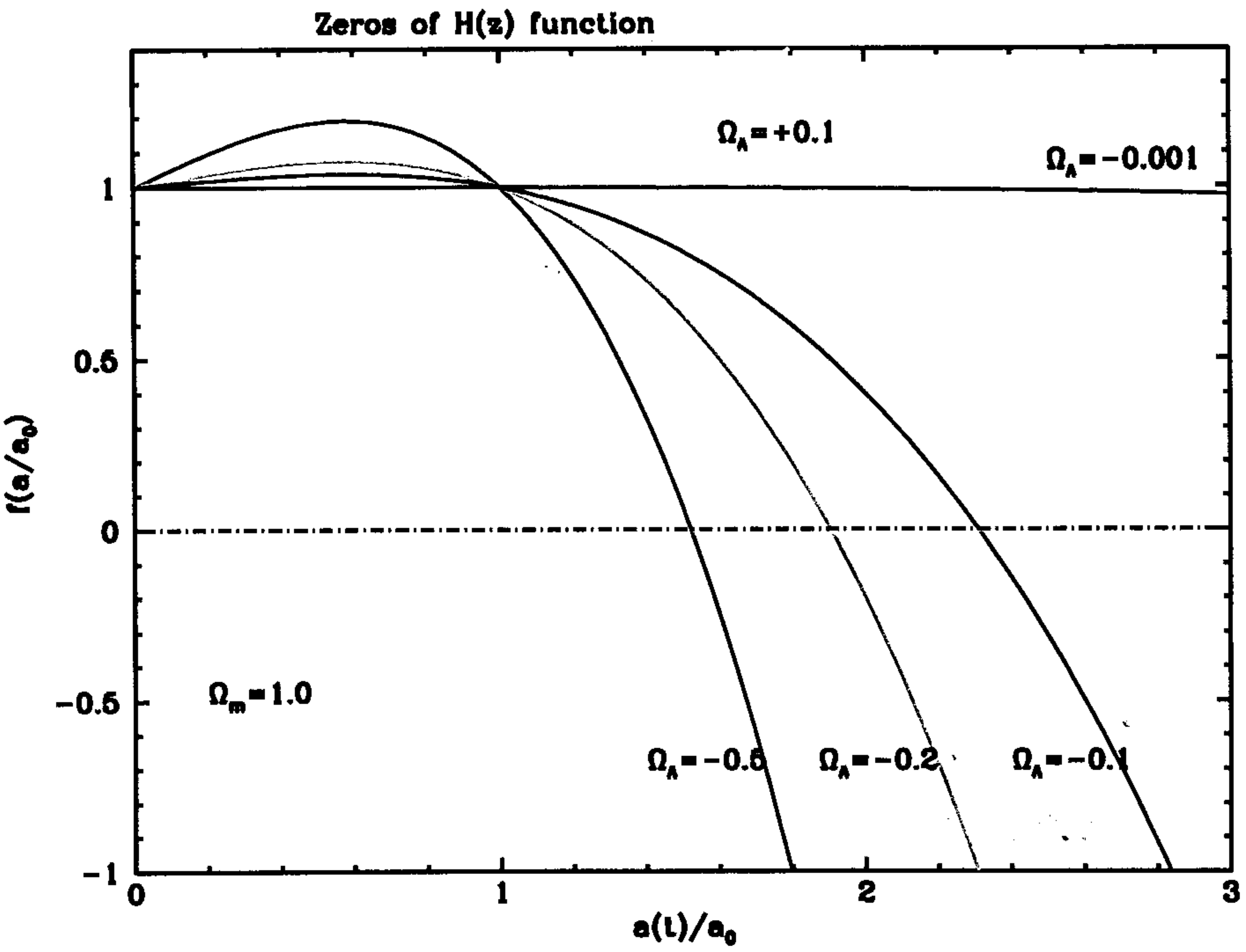
where $x = a(t)/a_0$ and $\Omega_K = 1 - \Omega_m - \Omega_R - \Omega_\Lambda$

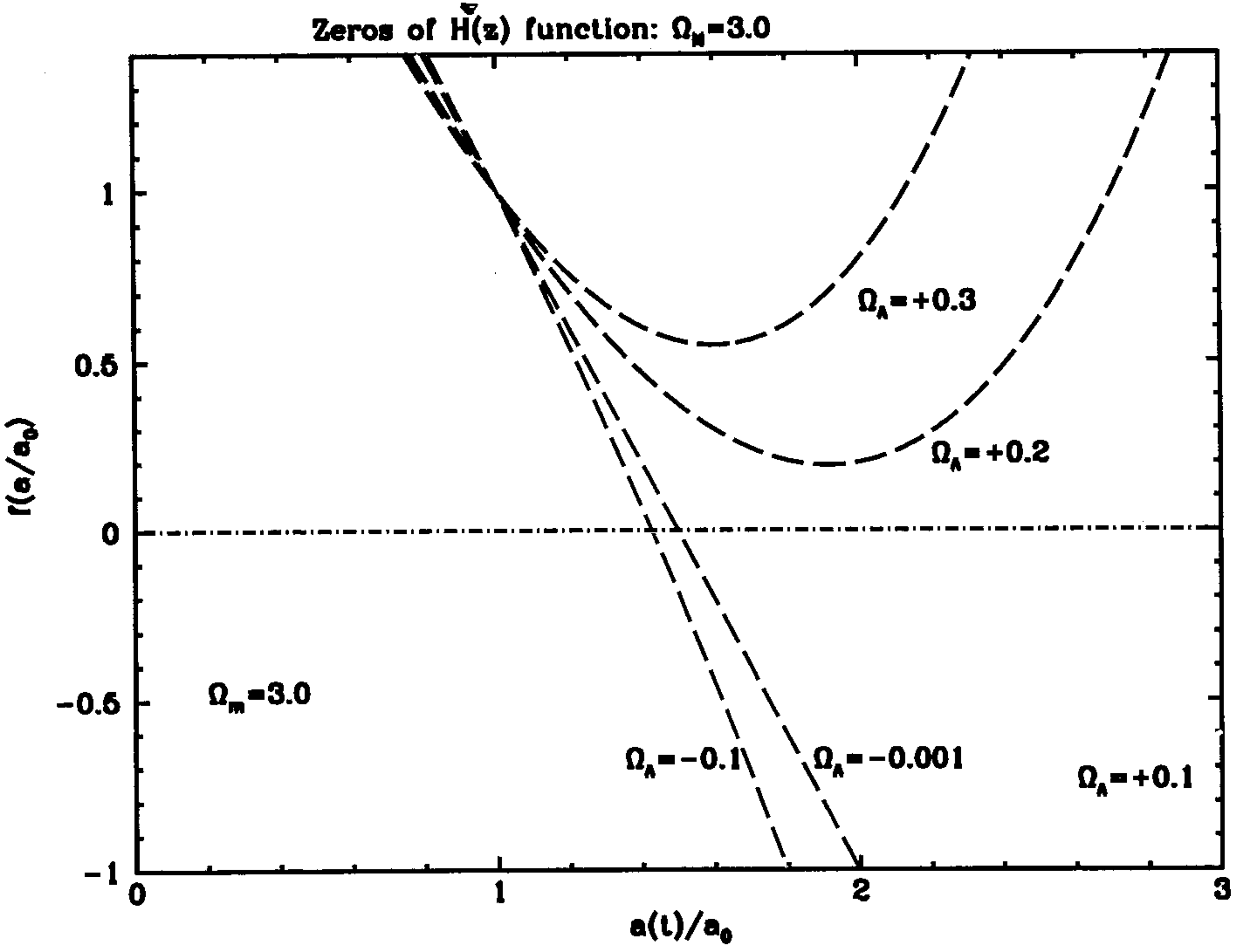
Figure shows example

Conclude:

(1) As Ω_m increases, recollapse occurs with increasing Ω_Λ. But rule of thumb $\Omega_\Lambda > 0 \Rightarrow$ forever

(2) If $\Omega_\Lambda = 0$, recollapse if $\Omega_m > 1$, closed


Zeros of H(z) function
$\Omega_\Lambda=+0.1$
$\Omega_\Lambda=-0.001$
$\Omega_m=1.0$
$\Omega_\Lambda=-0.5$
$\Omega_\Lambda=-0.2$
$\Omega_\Lambda=-0.1$
$f(a/a_0)$
$a(t)/a_0$
1
0.5
0
−0.5
−1
0
1
2
3



Zeros of $\tilde{H}(z)$ function: Ω_M=3.0
Ω_Λ=+0.3
Ω_Λ=+0.2
Ω_m=3.0
Ω_Λ=−0.1
Ω_Λ=−0.001
Ω_Λ=+0.1
f(a/a$_0$)
a(t)/a$_0$
1
0.5
0
−0.5
−1
0
1
2
3

Correction to flatness discussion.

Recall from eq. (13):

$$\Omega(t) - 1 = \left(\frac{H_0}{H(t)}\right)^2 \left(\frac{a_0}{a}\right)^2 (\Omega - 1)$$

$$H(t)^2 = \frac{8\pi G}{3}\rho - \frac{c^2 K}{a^2} \quad \text{or as we saw.}$$

$$\left(\frac{H(t)}{H_0}\right)^2 = \Omega_\Lambda + \Omega_m (1+z)^3 + \Omega_R (1+z)^4 + (1 - \Omega_\Lambda - \Omega_m)(1+z)^2$$

at earliest times $\Omega_R (1+z)^4$ dominates but this term is negligible today!

For convenience ~~let~~ let's assume $\Omega_\Lambda = .2$, $\Omega_M = .3$

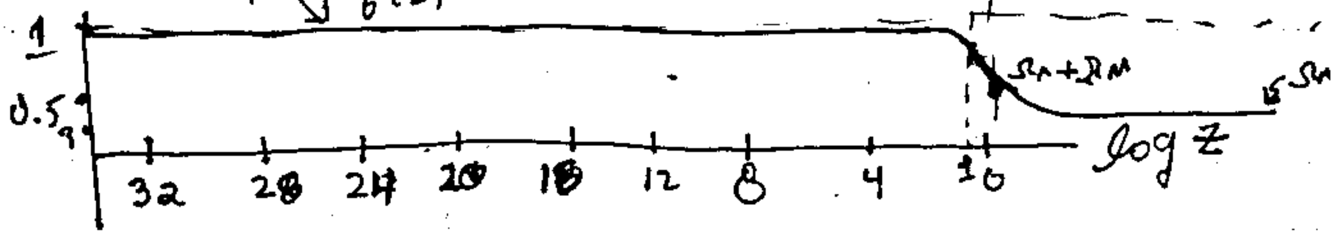
Recall $\Omega_R = 2.44 \times 10^{-5} h^{-2} \Rightarrow h = .5$

$$\Omega(t) - 1 = \frac{(1+z)^2 \, (\Omega - 1)}{\Omega_\Lambda + \Omega_m (1+z)^3 + \Omega_R (1+z)^4 + (1 - \Omega_\Lambda - \Omega_m)(1+z)^2}$$

$$\boxed{\Omega(t) - 1 = \frac{\Omega - 1}{\frac{\Omega_\Lambda}{(1+z)^2} + \Omega_m (1+z) + \Omega_R (1+z)^2 + (1 - \Omega_\Lambda - \Omega_m)}} \quad (2)$$

$$\lim_{z \to \infty} \Omega(t_p) - 1 = \frac{\Omega - 1}{\Omega_R (1+z)^2} = \frac{4.1 \times 10^4 h^2}{(1+z)^2} (\Omega - 1) \quad \text{(worked out last week)}$$

from eq. (2)



Only recently did $\Omega(t)$ deviate from unit. In the future Ω_Λ term dominates as $\lim_{\frac{a_0}{a_0}\to\infty} \Omega = \Omega_\Lambda \simeq .3$

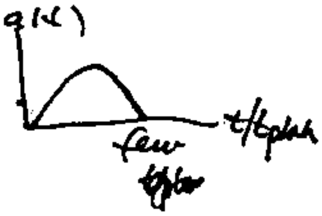
Finally suppose $\Omega(t_{planck}) = 1.1$

Recall $1+z_p \simeq 3\times10^{31}$

$$\Omega(t_{planck}) - 1 = \frac{4.07\times10^{4}h^2}{(1+z_{planck})^2}(\Omega-1)$$

$$\Rightarrow \Omega - 1 = \frac{\overbrace{(0.1)}^{\Omega(t_{planck})-1}(1+z_{planck})^2}{4.07\times10^{4}h^2} \simeq \frac{2\times10^{57}}{h^2}\ !$$

Price to pay is enormous: lifetime of such a Universe would be $\simeq$ few $\times\, t_{planck}$.
Only last about $\simeq 10^{-42}$ seconds!.!



Let's go back to lightlike geodesics: We need to compute them to determine observationally relevant distances, which involve measurements along our past lightcone

Geometry: FRW Metric Again -

$$ds^2 = c^2dt^2 - a^2(t)dl^2 ; \; dl^2 = a^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega\right]$$

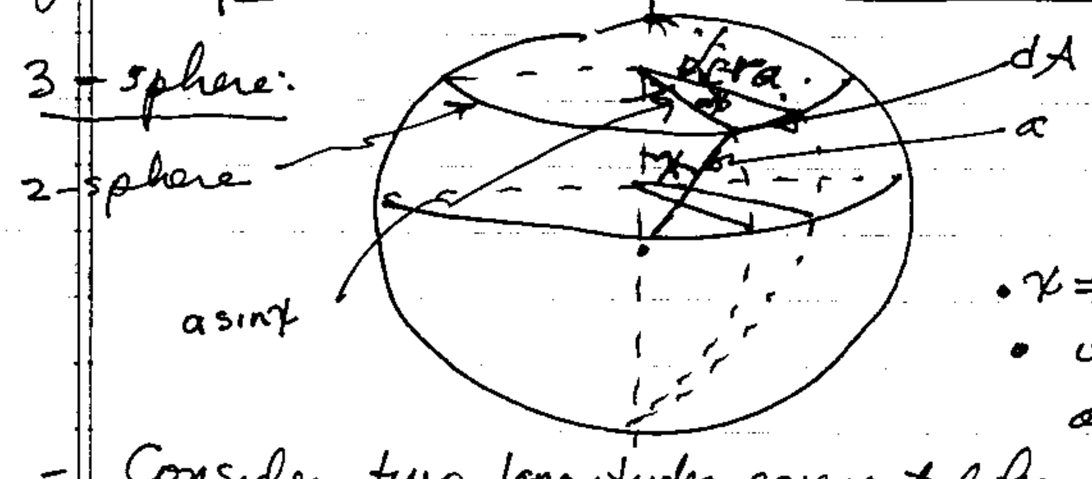
Closed Geometry: $k = +1$

Spatial part of metric: $dl^2 = a^2\left[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$

Introduce "polarangle" variable χ such that
$r = \sin\chi$; $dr = \cos\chi\, d\chi$; $1 - r^2 = \cos^2\chi$

Therefore: $\boxed{dl^2 = a^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]}$

3-sphere:



suppressed θ (in figure)

$a = a\sin\chi$

• r is "cylindrical" radius $a\sin\chi$

• $\chi = 0$ is origin

• $0 \le \chi \le \pi$ ("polar" angle on 3 sphere)

- Consider two longitudes separated by $d\phi$ (Figure suppresses θ)

proper area: $dA = a^2 r^2 d\Omega = a^2 \sin^2\chi \, \overbrace{\sin\theta\, d\phi\, d\theta}^{d\Omega}$

Integrate over 4π sterad $\boxed{A = 4\pi a^2 \sin^2\chi}$

Flat Geometry: $K=0$

$$dl^2 = a^2 [dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2]$$

Let $r = \chi$

Obviously $A = 4\pi a^2 \chi^2$

$\therefore 0 \le \chi \le \infty$

Open Geometry: $K=-1$

$$dl^2 = a^2 \left[\frac{dr^2}{1+r^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]$$

Let $r = \sinh(\chi)$; $dr = \cosh(\chi)$

In this case $dl^2 = a^2 [d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2)]$

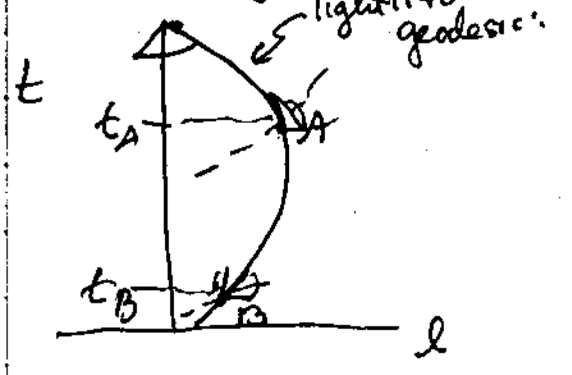
$$A(\chi) = 4\pi a^2 \sinh^2 \chi$$

Summarize : $ds^2 = c^2 dt^2 - a^2 [d\chi^2 + F^2(\chi) d\Omega]$

K	$F(\chi)$	$A(\chi)$	Range
$+1$	$\sin\chi$	$4\pi a^2 \sin^2\chi$	$0 \le \chi \le \pi$
0	χ	$4\pi a^2 \chi^2$	$0 \le \chi \le \infty$
-1	$\sinh\chi$	$4\pi a^2 \sinh^2\chi$	$0 \le \chi \le \infty$

$$dl^2 = a^2 [d\chi^2 + F^2(\chi)(d\theta^2 + \sin^2\theta \, d\phi^2)]$$

Distance: Ambiguous



Although FO B is further back in time, its proper radial distance is smaller than proper distance to A at t_A

$$d\ell = a\, d\chi \Rightarrow \ell(t) = a(t)\,\chi \quad \left\{\begin{array}{l}\text{proper distance}\\ \text{at } t\end{array}\right.$$

But we cannot measure $\ell(t)$. Instead let's use empirical measures of distance!

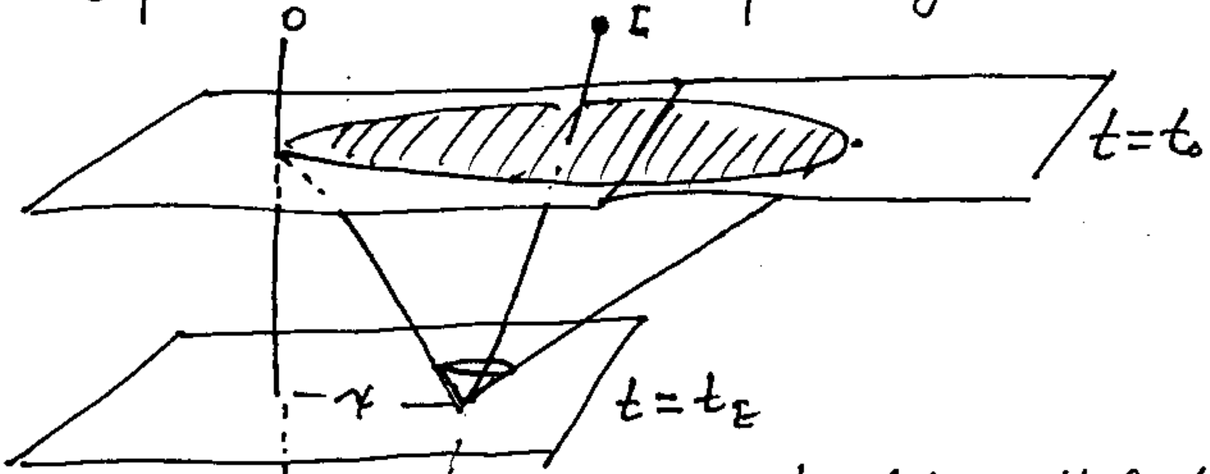
<u>Luminosity distance:</u>

Photon emitted at $t_E(z)$ with frequency ν detected at $t_0 = t(z=0)$ with frequency ν_0.

$$\text{Flux} = S_{\nu_0} d\nu_0 = \frac{L_\nu d\nu}{4\pi d_L^2} \quad : \text{defines } d_L$$

Photons emitted at $\nu, \nu + d\nu$ detected at $\nu_0, \nu_0 + d\nu_0$

Assume photons emitted isotropically:



Light emitted at t_E spreads out such that at t_0, it spreads over a 2-sphere with area $A = 4\pi a_0^2 F^2(\chi)$ where χ is comoving radial coordinate of emitter E.

As a result: $S_{\nu_0} d\nu_0 = \frac{L_\nu d\nu}{4\pi (a_0 F(\chi))^2} \times \frac{1}{(1+z)^2}$

Extra $(1+z)^{-2}$ arises because

(a) photon energy received less than emitted: $h\nu_0 = h\nu_E/(1+z)$

(b) Arrival time interval gets stretched: $dt_0 = (1+z)\, dt_E$

Definition of flux: $\text{Flux} = \frac{\#\text{ photons detected} \times \text{photon energy}}{\text{Area} \cdot \text{time interval}}$

$\therefore \; S_{\nu_0} d\nu_0 = \frac{\Delta N \cdot h\nu_0}{\text{Area} \cdot \Delta t_0} = \frac{\Delta N \cdot h\nu (1+z)^{-1}}{\text{Area} \cdot (1+z) dt_E}$

$\Delta N = \#$ photons emitted in $(\nu, \nu+d\nu)$ = # photons detected in $(\nu_0, \nu_0+d\nu_0)$

$S_{\nu_0} d\nu_0 = \left[\frac{\Delta N \cdot h\nu}{dt_E}\right] \times \frac{1}{\text{Area} \cdot (1+z)^2}$

$S_{\nu_0} d\nu_0 = \frac{L_\nu d\nu}{\text{Area} \times (1+z)^2} = \frac{L_\nu d\nu}{4\pi (a_0 F(\chi))^2 (1+z)^2}$

$\boxed{d_L = a_0 F(\chi)(1+z)}$ defines luminosity distance

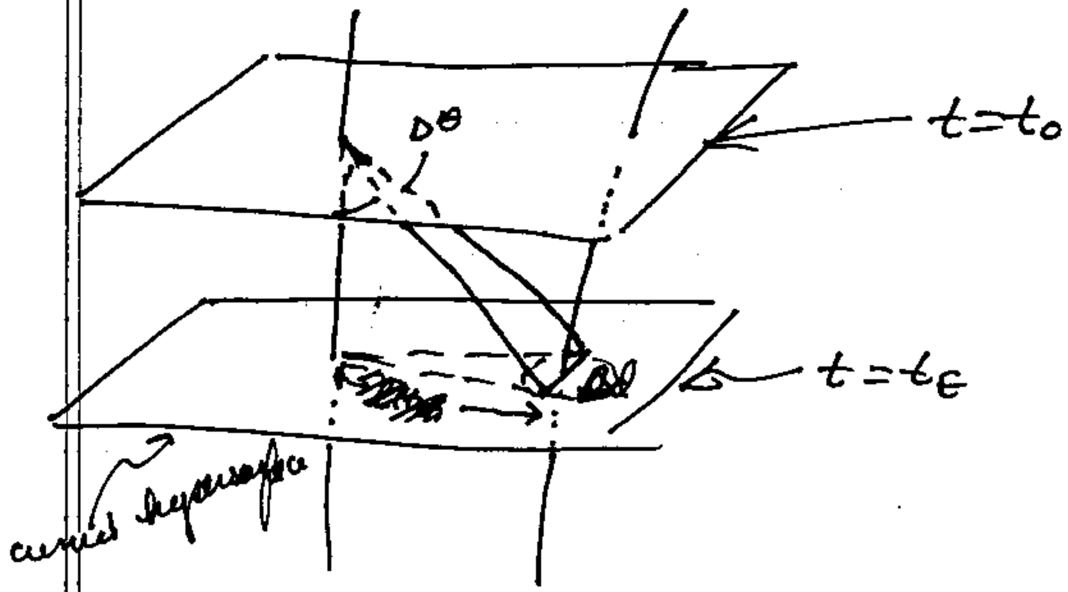
$S_{\nu_0} d\nu_0 = \frac{L_\nu d\nu}{4\pi d_L^2} \Rightarrow S_{\nu_0} = (1+z) \frac{L_\nu}{4\pi d_L^2}$

Distance by apparent size or angular diameter distance

Suppose we have independent determination of the linear size of some distant object.

- Could be size of galaxy-
- "Sound" horizon..

Let $d_A = \frac{\Delta l}{\Delta \Theta}$ where $\Delta\Theta$ = angular size!



Consider two light like geodesics ~~separated~~
- detected at $t=t_0$
- separated by $\Delta\theta$, but $\Delta\phi=0$
- Source has proper linear diameter Δl at $\underline{\text{fixed}}$ comoving coordinate χ

$\Delta d = dl = \underbrace{a(t_E) F(\chi)\,d\theta}$

Clearly $\Delta l = a(t_E) \times F(\chi)\cdot\Delta\theta$ on ~~[illegible]~~

As a result:

$$d_A = \frac{\Delta l}{\Delta\theta} = a(t_E)F(\chi)$$

or

$$d_A = \frac{a(t_E)}{a(t_0)} \times a(t_0)F(\chi)$$

$$d_A = \frac{a(t_0)F(\chi)}{(1+z)}$$

Comparison with definition of d_L shows that $\boxed{d_L = (1+z)^2 d_A}$

But to get either ~~given~~ distance as a function of z we need to integrate radial lightlike geodesics along our

past light-cone

Simple case first: Einstein- de Sitter Universe

$k=0, \ \Omega_\Lambda=0, \ \Omega_M=1$ (Ignore Ω_R)

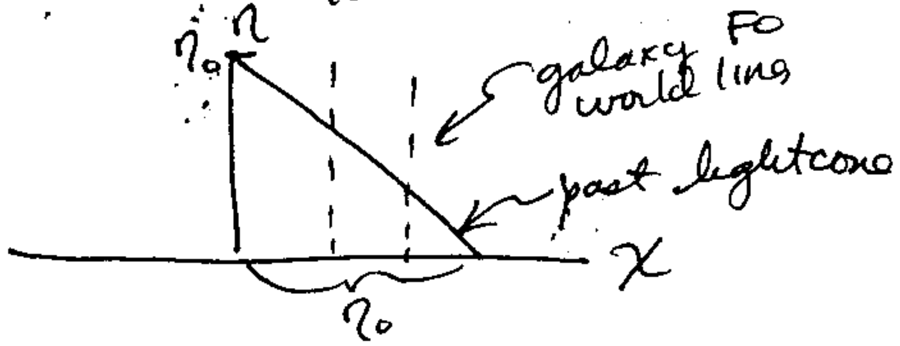
Radial Light-like Geodesics

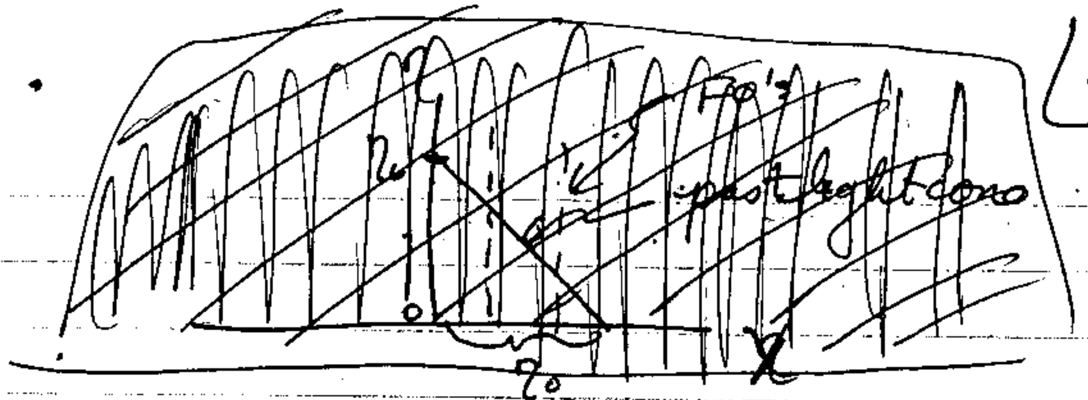
Recall $ds=0 \Rightarrow d\chi = -\dfrac{c\,dt}{a(t)}$: incoming

Define conformal time : $d\eta \equiv \dfrac{dt}{a(t)}$

As a result.. incoming radial photons $d\chi = -d\eta$

$$\int_0^{\chi} d\chi = -\int_{\eta_0}^{\eta} d\eta' = \eta_0 - \eta \Rightarrow \chi(\eta) = \eta_0 - \eta$$





Einstein-de Sitter Model ($\Omega_m = 1, \Omega_\Lambda = 0, \Omega_K = 0$)

Recall $a(t) = a_0 (t/t_0)^{2/3}$; $t_0 = \frac{2}{3} H_0^{-1}$

$$\chi(t) = -c\int_{t_0}^{t(z)} \frac{dt}{a_0} \frac{t_0^{2/3}}{t^{2/3}} = \frac{-c t_0^{2/3}}{a_0} \left[\frac{t^{1/3}}{1/3} \right]_{t_0}^{t(z)}$$

$$\chi(t) = \frac{-3c t_0^{2/3}}{a_0} \left[t^{1/3} - t_0^{1/3} \right] = \frac{3 c t_0}{a_0} \left[1 - \left(\frac{t}{t_0}\right)^{1/3} \right]$$

But $\left(\frac{t}{t_0}\right)^{1/3} = \left(\frac{a}{a_0}\right)^{1/2} = \frac{1}{(1+z)^{1/2}}$

As a result: $\chi(t) = 3\left(\frac{2}{3}\frac{c}{H_0}\right)\left(\frac{1}{a_0}\right)\left[1 - \frac{1}{(1+z)^{1/2}}\right]$

$$\chi(z) = 2\left(\frac{c}{H_0}\right)\left(\frac{1}{a_0}\right)\left[1 - (1+z)^{-1/2}\right]$$

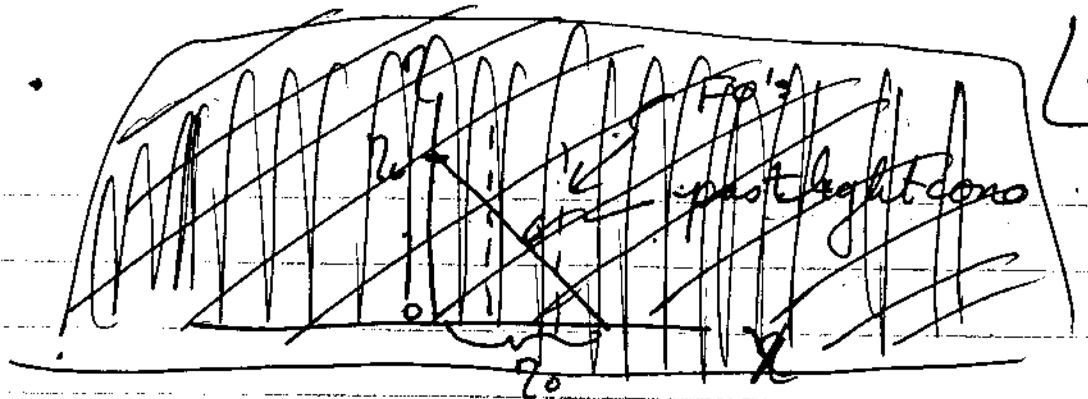
Distance by apparent Size:

$$d_A = a(t_E)\chi(z) = \frac{a_0}{(1+z)} \times \chi(z)$$

Therefore

$$d_A = \frac{a_0}{(1+z)} \times 2\left(\frac{c}{H_0}\right)\left(\frac{1}{a_0}\right)\left[1 - (1+z)^{-1/2}\right]$$

$$\boxed{d_A = 2\left(\frac{c}{H_0}\right)(1+z)^{-1}\left[1 - (1+z)^{-1/2}\right]}$$



Einstein-de Sitter Model $(\Omega_m = 1, \Omega_\Lambda = 0, \Omega_K = 0)$

Recall $a(t) = a_0 (t/t_0)^{2/3}$; $t_0 = \frac{2}{3} H_0^{-1}$

$$\chi(t) = -c\int_{t_0}^{t(z)} \frac{dt}{a_0} \frac{t_0^{2/3}}{t^{2/3}} = \frac{-c t_0^{2/3}}{a_0} \left[\frac{t^{1/3}}{1/3} \right]_{t_0}^{t(z)}$$

$$\chi(t) = \frac{-3c t_0^{2/3}}{a_0} \left[t^{1/3} - t_0^{1/3} \right] = \frac{3ct_0}{a_0} \left[1 - \left(\frac{t}{t_0}\right)^{1/3} \right]$$

But $\left(\frac{t}{t_0}\right)^{1/3} = \left(\frac{a}{a_0}\right)^{1/2} = \frac{1}{(1+z)^{1/2}}$

As a result: $\chi(t) = 3\left(\frac{2}{3}\frac{c}{H_0}\right)\left(\frac{1}{a_0}\right)\left[1 - \frac{1}{(1+z)^{1/2}}\right]$

$$\chi(z) = 2\left(\frac{c}{H_0}\right)\left(\frac{1}{a_0}\right)\left[1 - (1+z)^{-1/2}\right]$$

Distance by apparent Size:

$$d_A = a(t_z)\chi(z) = \frac{a_0}{(1+z)} \times \chi(z)$$

Therefore

$$d_A = \frac{a_0}{(1+z)} \times 2\left(\frac{c}{H_0}\right)\left(\frac{1}{a_0}\right)\left[1 - (1+z)^{-1/2}\right]$$

$$\boxed{d_A = 2\left(\frac{c}{H_0}\right)(1+z)^{-1}\left[1 - (1+z)^{-1/2}\right]}$$

Key Feature: $d_A(z)$ has a maximum value at finite redshift.

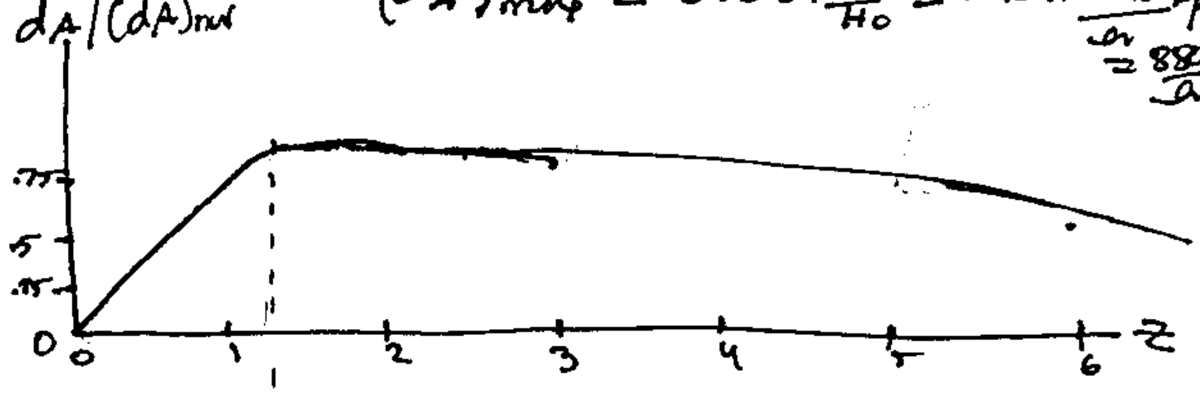
$$\frac{d_A}{dz} = \text{const} \times \frac{d}{dz}\left\{(1+z)^{-1} - (1+z)^{-3/2}\right\} = 0$$

$$-(1+z)^{-2} + \frac{3}{2}(1+z)^{-5/2} = 0$$

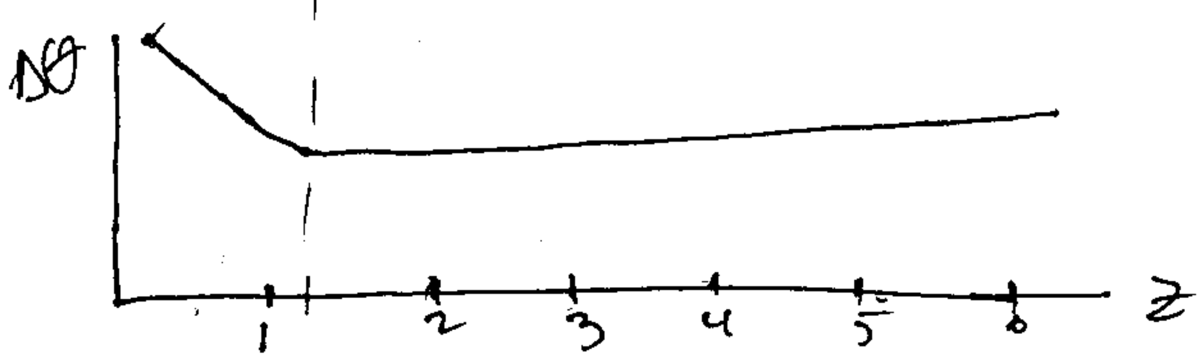
$$(1+z)^{1/2} = \frac{3}{2} \Rightarrow 1+z_{max} = \frac{9}{4}$$

$$\Rightarrow z_{max} = \frac{5}{4} = 1.25$$

$$(d_A)_{max} = 0.296 \times \frac{c}{H_0} = .296 \times \frac{3\times10^3}{h} \text{Mpc} = \frac{880}{h} \text{Mpc}$$

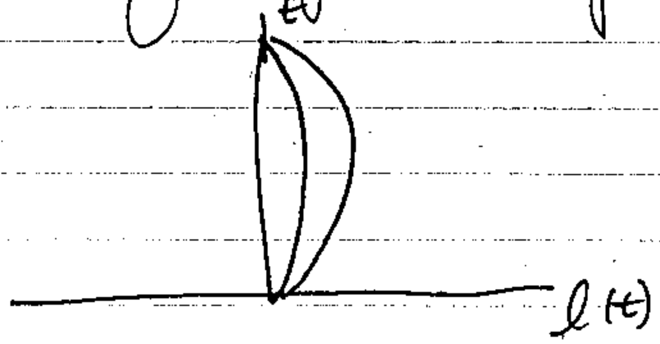


d_A decreases very slowly as z increases for $z > z_{max}$. As a result the angular diameters have minima & increase for $z > z_{max}$



⊖ very insensitive to z for large range in redshifts

This is a generic property of Friedmann models and is related to spacetime curvature of lightlike geodesics!



Now let's look at generic case

past lightcone:

along radial geodesics $d\theta = d\phi = 0$.

$\therefore \; ds^2 = c^2 dt^2 - a^2(t) d\chi^2 = 0$

$$\text{or} \quad d\chi = -\frac{c\,dt}{a(t)}$$: inward progating light

Let $a(t) = x \cdot a_0 \quad \Rightarrow \quad d\chi = -\dfrac{c\,dt}{a_0 x}$

Recall: $$dt = \frac{dx}{H_0 x \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

$$\therefore \quad d\chi = -\frac{c\,dx}{a_0 H_0 x^2 \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

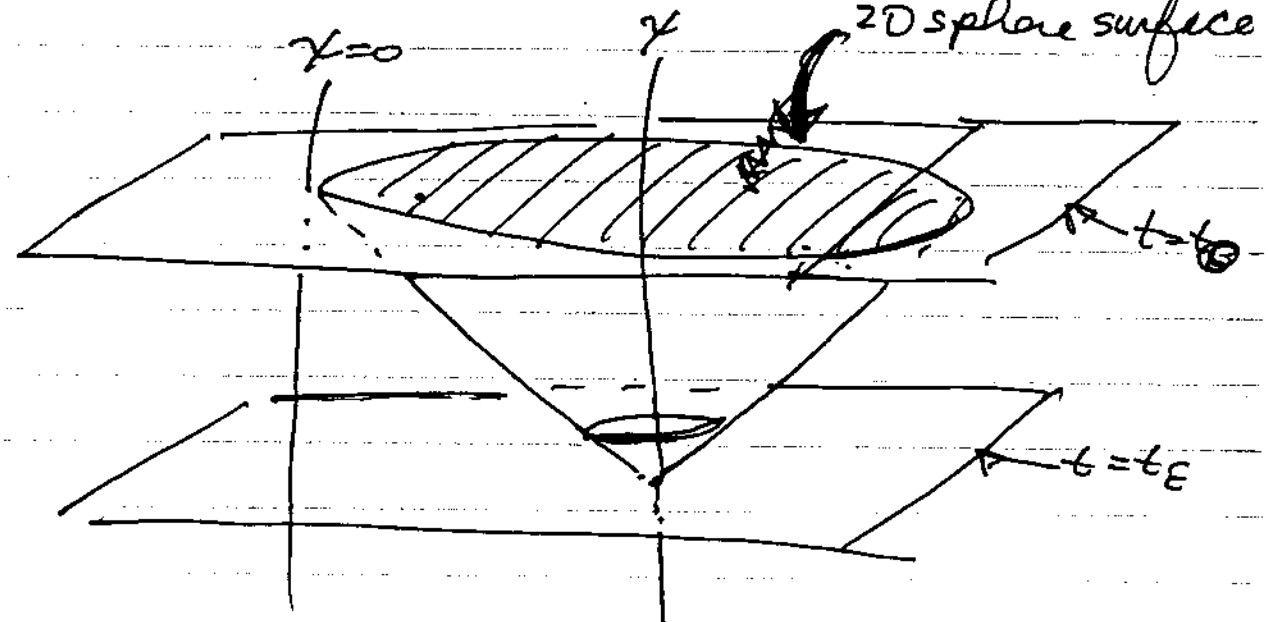
$$\chi(z) = \int_0^{\chi(z)} d\chi = \frac{-c}{a_0 H_0} \int_1^{1/1+z} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

$$\text{or} \quad \chi(z) = \left(\frac{c}{a_0 H_0}\right) \int_{\frac{1}{1+z}}^{1} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

Now get expressions for d_A, d_L:

Recap — Distance

(1) Luminosity Distance:



Emitter generates forward light cone: Expanding sphere of radiation. Emitted at t_E, then detected by us at $t=t_0$. We worked out area of sphere surrounding us at $t=t_0$ with comoving radial coordinate χ. But we can do the same for sphere surround FO at χ since in his frame we ($\chi=0$) are at comoving coordinate χ relative to this FO: the two areas are equivalent..

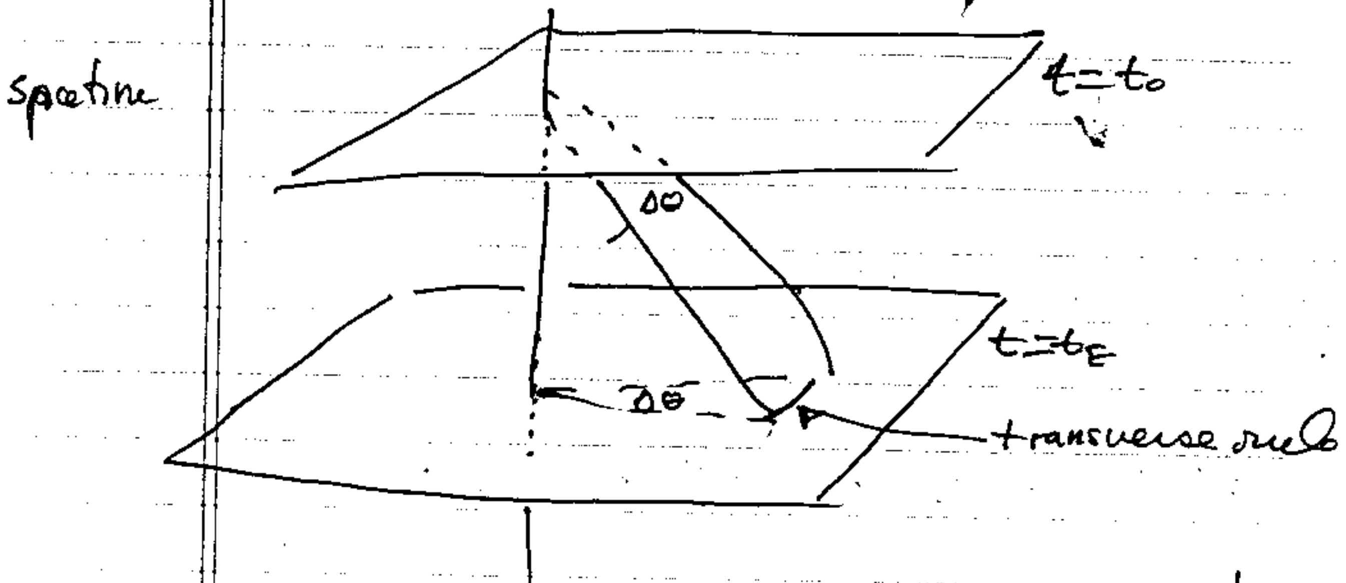
$$\boxed{A(t_0) = 4\pi a_0^2 F^2(\chi)}$$

$$S_{\nu_0} d\nu_0 = \frac{L_\nu d\nu}{A(t_0)} \times \frac{1}{(1+z)^2} \Rightarrow \frac{L_\nu d\nu}{4\pi d_L^2}$$

$$d_L = a_0 F(\chi)(1+z)$$

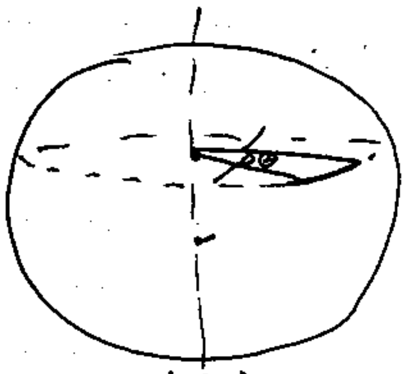
(2) Distance by apparent Size

Here we measure distance by comparing angular diameter of an object to its transverse linear diameter



But geometry [is that of a] $t=t_E$ hypersurface

$$dl^2 = a^2[d\chi^2 + F^2(d\theta^2 + \sin^2\theta\, d\phi^2)]$$



across ruler $d\chi = d\phi = 0$

$\therefore\ \Delta d = dl = a(t_E)F(\chi)\Delta\theta$

Since $d_A \equiv \dfrac{\Delta d}{\Delta\theta} = a(t_E)F(\chi) = \dfrac{a_0 F(\chi)}{(1+z)}$

differs from Euclidean result.

Note d_A is not proper radial distance between 2 FO's with different values of χ.

In that case $d\theta = d\phi = 0$. So

Valid for any K $\begin{cases} dl = a(t)d\chi \\ l(t_E) = a(t_E)\chi \end{cases}$ $\left\{ \begin{array}{c} l(t_E) = d_A \text{ only} \\ \text{in } K=0 \text{ case} \end{array} \right\}$

In $K \neq 0$ models $l(t_E) \neq d_A(t_E)$

Back to General Case

Last time I showed that comoving radial coordinate of object detected along our past lightcone at redshift z is given by integrating lightlike geodesics:

$$d\chi = -\frac{c\,dt}{a(t)}$$

We found that

$$\chi(z) = \left(\frac{c}{a_0 H_0}\right) \int_{(1+z)^{-1}}^{1} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}$$

Recall: $\Omega_K = \frac{-c^2 K}{(a_0 H_0)^2} \Rightarrow \left(\frac{c}{a_0 H_0}\right) = \sqrt{\frac{\Omega_K}{-K}}$

$$\Omega_K = \sqrt{\frac{1-\Omega_\Lambda-\Omega_M-\Omega_R}{-K}} = \sqrt{\text{abs}(1-\Omega_\Lambda-\Omega_M-\Omega_R)}$$

Distance:

$$d_A = \frac{a_0 F[\chi(z)]}{1+z} =$$

$$\frac{c}{H_0}\sqrt{\frac{-K}{\Omega_K}} \times \frac{1}{(1+z)} \times F\left\{ \underbrace{\sqrt{\frac{\Omega_K}{-K}} \int_{(1+z)^{-1}}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_M x^{-3} + \Omega_R x^{-4}}}}_{\chi(z)} \right\}$$

Special Case: $K = 0$

In this case $F(\chi) = \chi$, and $\sqrt{\frac{\Omega_K}{-K}}$ term cancels, i.e.,

$$d_A = \left(\frac{a_0}{1+z}\right) \times \left\{ \frac{c}{a_0 H_0} \int_{(1+z)^{-1}}^{1} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \cancel{\Omega_K x^{-2}} + \Omega_m x^{-3} + \Omega_R x^{-4}}} \right\}$$

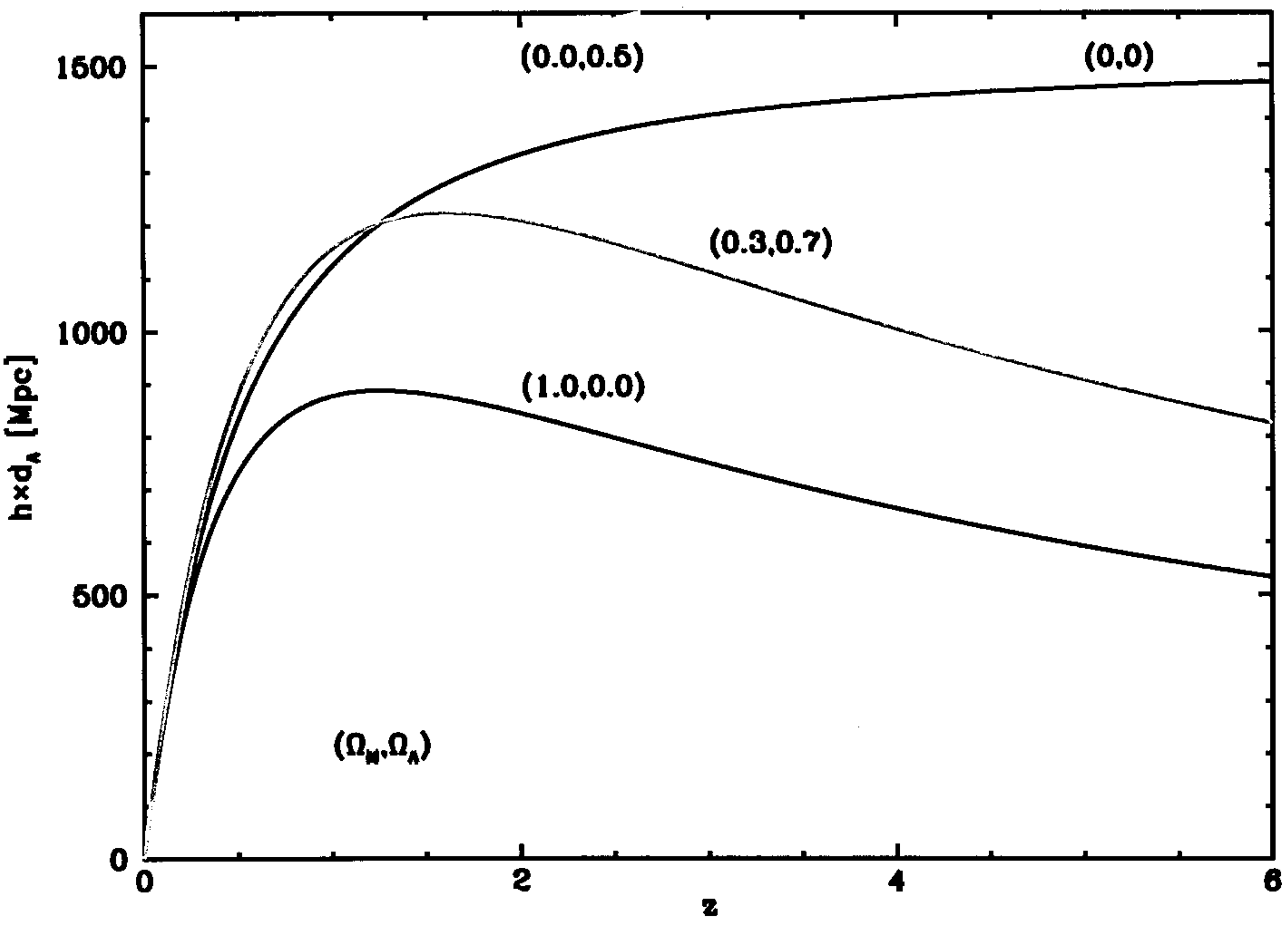
or $$d_A = \left(\frac{c}{H_0}\right)\left(\frac{1}{1+z}\right) \int_{(1+z)^{-1}}^{1} \frac{dx}{x^2 \sqrt{\Omega_\Lambda + \Omega_m x^{-3} + \Omega_R x^{-4}}}$$

Figure comparison of $d_A(z)$ for various cosmologies

(I) Fig. 1: d_A versus z

- Models with $\Omega_m \neq 0$ all exhibit maximum in d_A versus z plane
- Models with $\Omega_m = 0$ do not exhibit this maximum.

Maxima related to spacetime focusing of lightlike geodesics

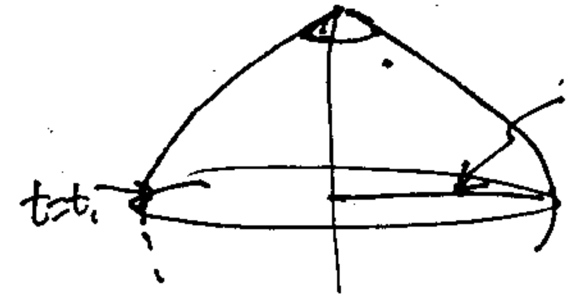


(0.0,0.5)
(0,0)
(0.3,0.7)
(1.0,0.0)
$(\Omega_M,\Omega_\Lambda)$
1500
1000
500
0
0
2
4
6
z
$h \times d_A$ [Mpc]

As result:

$$\left(\frac{8\pi G\rho}{3}\right)^{1/2}_{t} a(t_1) > \frac{c}{\chi(t_1)}$$

or $$(a(t)\chi)_{t_1} > \left(\frac{3c^2}{8\pi G\rho}\right)^{1/2}_{t_1}$$



$(a(t)\chi)_{t_1} = l(t_1)$: ~~proper radius of sphere resulting from light cone intersection $t=t_1$ hypersurface!~~

But $l(t_1)$ is just proper radius of sphere enclosed by past light cone:

Enclosed mass: $M(\chi_1) = \frac{4\pi}{3}\rho(t_1)a^3(t_1)\chi_1^3$

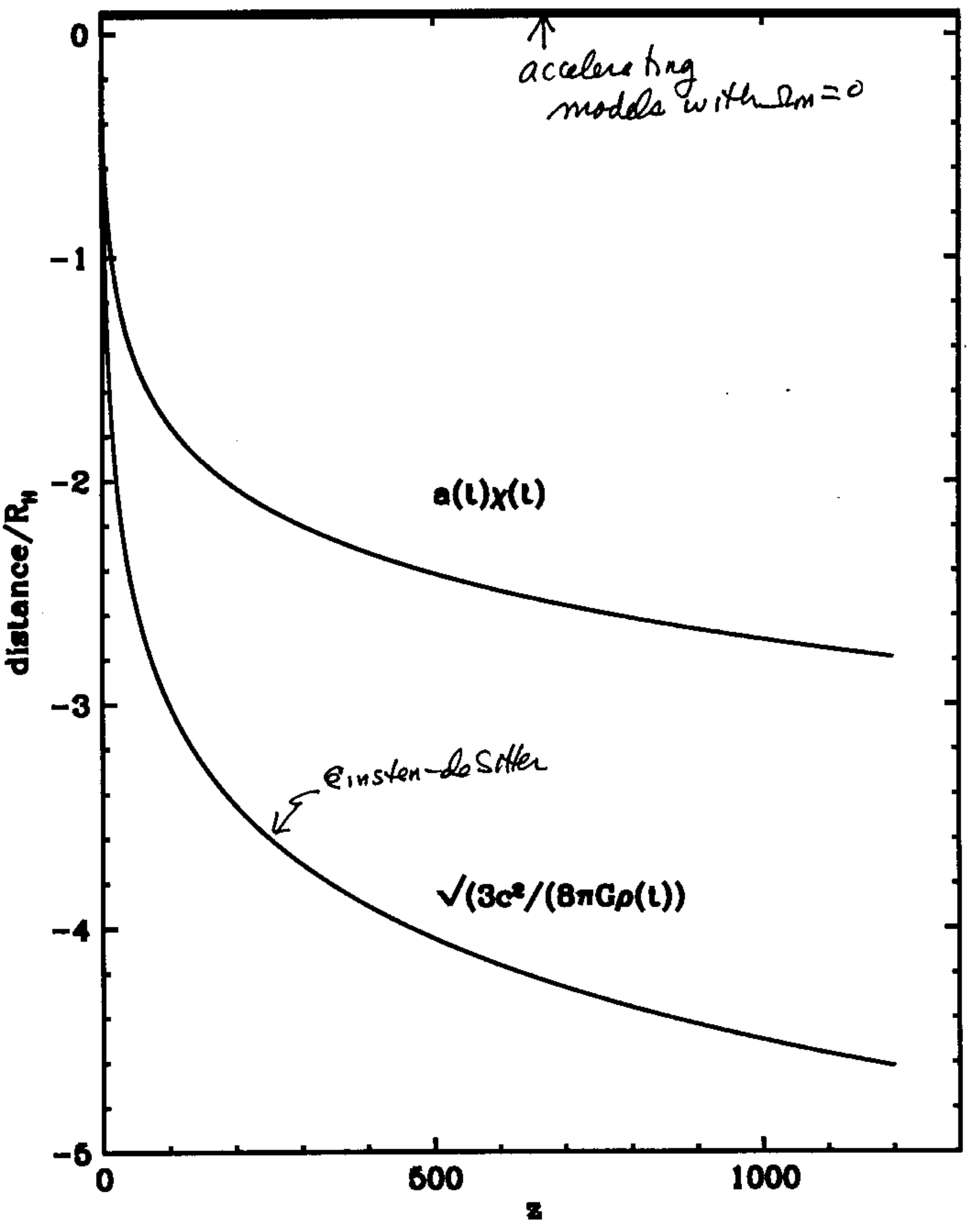
Schwarzschild radius $r_s = \frac{2GM}{c^2} = \frac{8\pi G\rho a^3\chi^3}{3c^2}$

But sphere will be in its Schwarzschild radius if

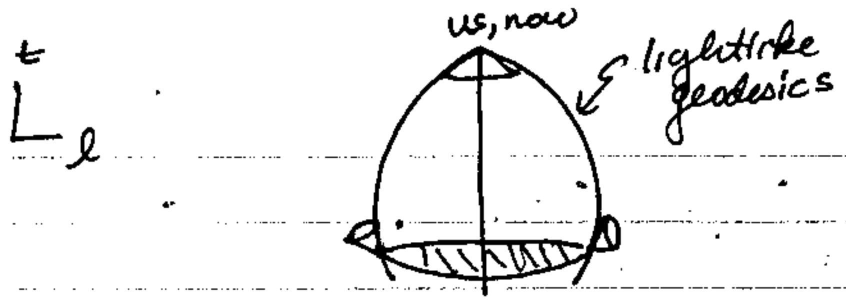
$$a(t_1)\chi(t_1) < r_s$$

$$a(t_1)\chi(t_1) < \frac{8\pi G\rho a^3\chi^3}{3c^2}$$

or if $(a\chi)^* > \left(\frac{3c^2}{8\pi G\rho}\right)^{1/2}$: convergence criterion!



accelerating models with Ωm=0
a(t)χ(t)
Einstein-de Sitter
√(3c²/(8πGρ(t))
distance/R_H
z
0
-1
-2
-3
-4
-5
0
500
1000



Past lightcone intersects a sphere on spacelike hypersurface $t = t_1$, with area

$$A(t) = 4\pi a^2(t) F^2(\chi(t))$$

where $\chi(t)$ is comoving radial coordinate of the sphere.

Geodesics converge provided:

$$\left(\frac{dA}{dt}\right)_{t_1} > 0 \quad \text{or} \quad \left[\frac{d}{dt}(a^2 F^2)\right]_{t_1} > 0$$

This implies: $2a\dot{a}F^2 + a^2 2F \frac{dF}{d\chi}\frac{d\chi}{dt} > 0$

Divide by $2a^2F^2$ and we have

$$\frac{\dot{a}}{a} > -\frac{1}{F}\frac{dF}{d\chi}\left(\frac{d\chi}{dt}\right)$$

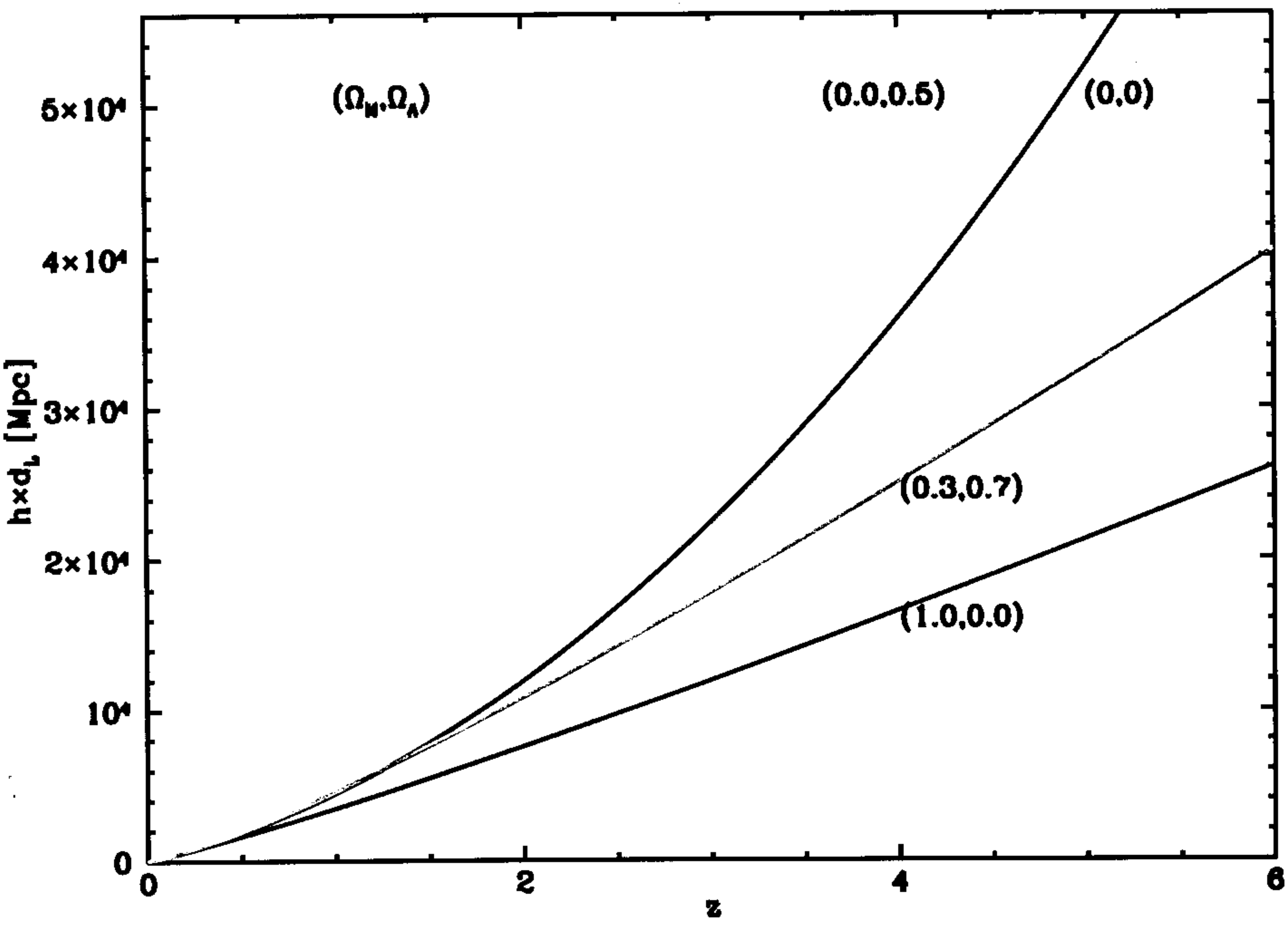
along the past light cone: $d\chi = -\frac{c\,dt}{a}$

$$\therefore \quad \frac{\dot{a}}{a} > \frac{1}{F}\frac{dF}{d\chi}\left(\frac{c}{a}\right)$$

Friedmann eq. $\Rightarrow$ $\dot{a} = \left(\frac{8\pi G\rho a^2}{3} - c^2K\right)^{1/2}$

Ignore curvature term in past so we then have

$$\dot{a} = \left(\frac{8\pi G\rho}{3}\right)^{1/2} a \;; \quad F(\chi) = \chi : \quad \frac{dF}{d\chi} = 1$$



$(\Omega_M,\Omega_\Lambda)$
(0.0,0.5)
(0,0)
(0.3,0.7)
(1.0,0.0)
$h\times d_L$ [Mpc]
5×10^4
4×10^4
3×10^4
2×10^4
10^4
0
2
4
6
z

(7.) Thus light rays along past lightcone diverge until ~~enclosed~~ mass enclosed by intersecting space-like 2-spheres is sufficiently large for convergence: these are called trapped surfaces.

(2) Luminosity distance

• No maxima { Show figure }
• Also differ

Models with $\Omega_\Lambda \neq 0$ have systematically larger $d_L(z)$ than models with $\Omega_m \neq 0$ and $\Omega_\Lambda = 0$.

(3) SN data.

(4.) SN data.

I discussed these standard candles before. Point is that we can obtain their luminosities or absolute magnitudes

$$S_{\nu_0} = \frac{L_\nu (1+z)}{4\pi d_L^2} = \frac{\overbrace{(L_\nu/L_{\nu_0})}^{r_\nu} L_{\nu_0}(1+z)}{4\pi d_L^2}$$

$$m_{AB} = -2.5 \log S_{\nu_0} - 48.6 \quad \left\{ \begin{matrix} \text{Defines AB} \\ \text{magnitude} \end{matrix} \right\}$$

$$\boxed{M_{AB} = m_{AB}(d_L = 10\,pc) = -2.5 \log\left(\frac{r_\nu L_{\nu_0}(1+z)}{4\pi (10)^2}\right) + 48.6}$$

To get absolute magnitude, compute flux that same object would yield if $d_L = 10\,pc$.

$$S_{\nu_0}(d=10) = \frac{L_{\nu_0}}{4\pi (10\,pc)^2} \quad : \text{since } z \ll 1.$$

$$m_{AB} - M_{AB} = -2.5 \log\left(\frac{S_{\nu_0}(d_L)}{S_{\nu_0}(10pc)}\right) = -2.5 \log\left[K_\nu (1+z) \left(\frac{10pc}{d_L}\right)^2\right]$$

Therefore $m_{AB} - M_{AB} = -2.5 \log[(1+z) K_\nu] + 5 \log\left(\frac{d_L}{10pc}\right)$

Standard: $m_{AB} - M_{AB} = K_\nu + \underbrace{5 \log(d/10pc)}_{\text{distance modulus}}$

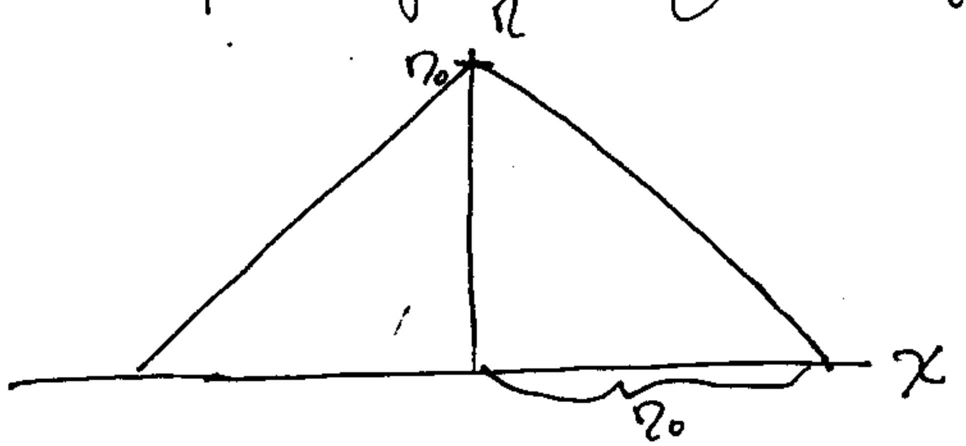
- K_ν is called the K correction and is well determined from spectra of SNIa
- M_{AB} deduced. So distance modulus can be obtained and compared with models.

Figure shows that $K=0$, $\Omega_m = .74$, $\Omega_\Lambda = 0.26$ is a good fit. $\Omega_m = 1$, $\Omega_\Lambda = 0$ is ruled out with high confidence. Good evidence that Universe is accelerating!

Horizons

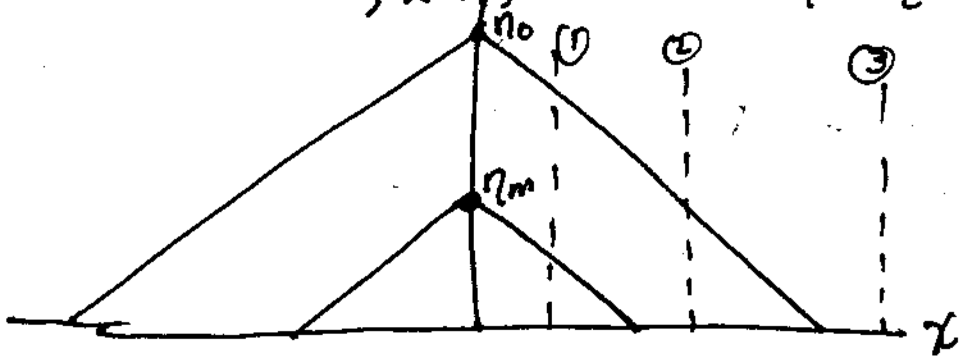
Back to conformal time: How much matter can we see?

Recall past light cone equation: $\chi = \eta_0 - \eta$



- at present time ($\eta = \eta_0$) we can in principle observe matter with comoving coordinates $\chi \leq \eta_0$

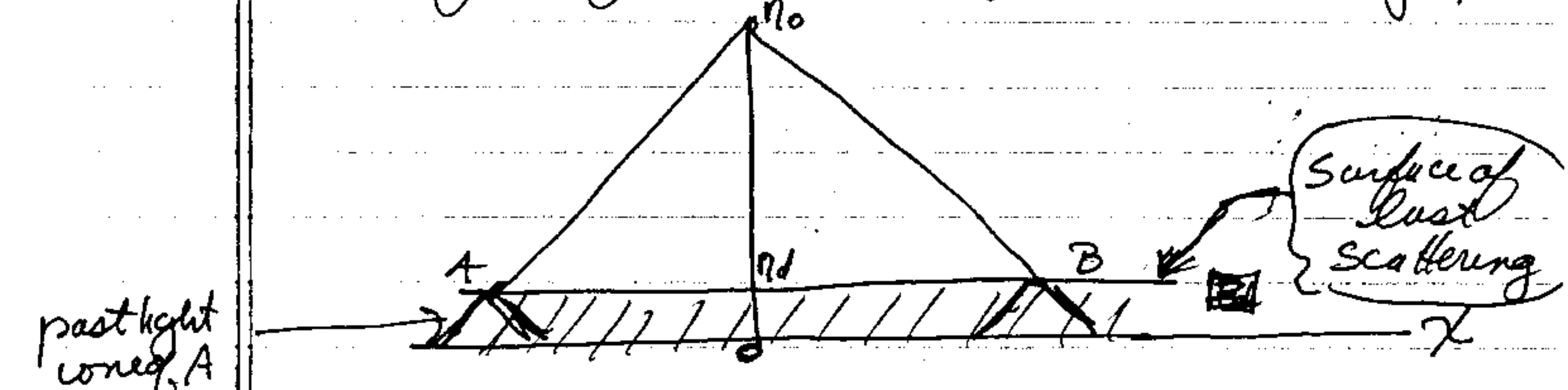
- at earlier time $\eta = \eta_m$, where $\eta_m < \eta_0$, we can see less; i.e., out to $\chi \leq \eta_m$



- at η_m: cannot detect 2 and 3: detect 1
- at η_0: cannot dect 3; detect 1 and 2

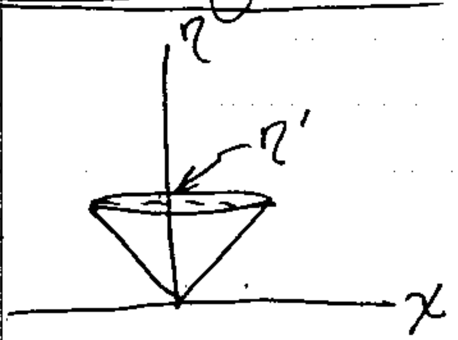
Horizon radius $\chi_H(\eta) = \eta$: Forward light con from $\eta = 0$: FO's come in horizon as η increase

<u>Causality</u>: As we shall see, CMB radiation emitted from surface of last scattering or decoupling epoch



Radiation propagates from $\eta = \eta_d$ hypersurface. Different parts of the sky have the same temperature, to one part in 10^5. This is a puzzle since past light cones of A & B don't overlap. Thus matter in two regions were not in causal contact according to FRW model.

<u>Horizon Equations</u>



(1) Horizon radius at $\eta = \eta'$ given by $\chi_H = \eta'$

(2) proper radius: $l_H = a(t)\chi_H$

$l_H(\eta) = a(\eta)\chi_H(\eta) = a(\eta)\eta$

Model dependence comes from form $a(\eta)$ or $a(t)$.

<u>Einstein-de Sitter Example</u> : $a(t) = a_0(t/t_0)^{2/3}$.

For physical scalar. Let $d\eta = \frac{c\,dt}{a(t)}$ $\left\{ \text{makes } \eta \text{ dimensionless} \right.$

$$\therefore\ \eta = c\int_0^t \frac{dt}{a_0(t/t_0)^{2/3}} = \frac{c t_0^{2/3}}{a_0}\int_0^t dt\, t^{-2/3}$$

$$\eta = \frac{3c\, t_0^{2/3} t^{1/3}}{a_0}$$

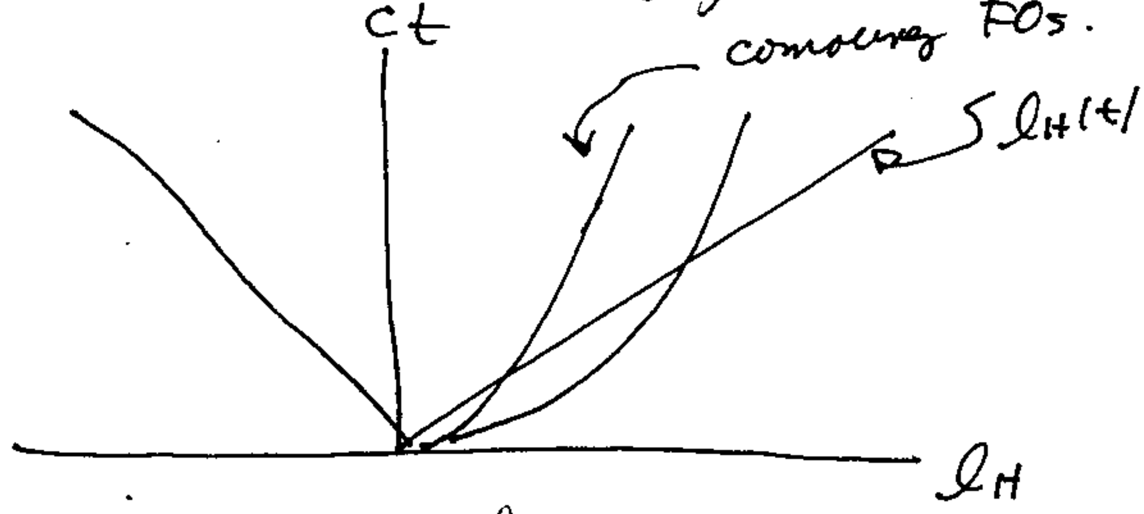
Recall: Proper horizon radius: $l_H(t) = a(t)\chi_H$

$$l_H = \cancel{a_0}\left(\frac{t}{t_0}\right)^{2/3} \frac{3c\,\cancel{t_0^{2/3}}}{\cancel{a_0}}\, t^{1/3}$$

$$\Rightarrow \boxed{l_H(t) = 3ct}$$

Proper distance to comoving observers: $l_H = a(t)\chi$ where $\chi = \text{const}$

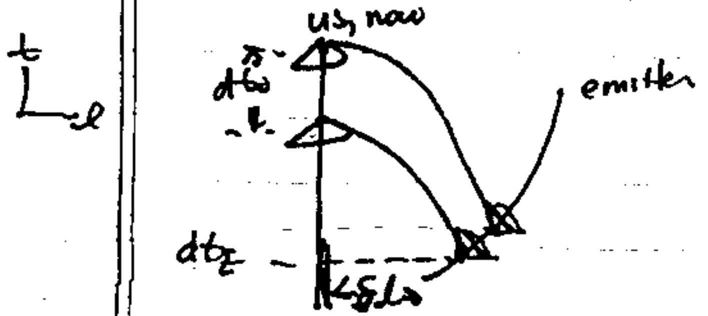
$$\Rightarrow\ l_H = a_0\left(\frac{t}{t_0}\right)^{2/3}\chi$$



Galaxies come into horizon as universe expands.

Radiative Transfer in an expanding Universe

Consider mutually expanding emitter and observer of radiation. If photons are emitted in time interval dt_E and observed in stretched time interval dt_o.



Intensity definition

$$I_\nu = \frac{dN \cdot h\nu}{dA_\perp \, \Delta\nu \, \Delta t \, d\Omega}$$

- Assume emitter & observer separated by infinitesmal proper distance δl. Then light-travel time $\delta t = \delta l / c$

- During trip photon suffers frequency shift
$$\delta\nu = -\nu \frac{u}{c} = -\nu \left[\frac{\dot a}{a} \delta l\right] / c = -\nu \frac{\dot a}{a} \delta t$$

- Frequency bandwidth changes:
$$\delta[\Delta\nu] = \Delta\nu[t+\delta t] - \Delta\nu(t)$$
$$\delta[\Delta\nu] = -\Delta\nu \frac{\dot a}{a} \delta t$$

- As a result $\frac{h\nu}{\Delta\nu} = \text{const}$ along lightlike geodesics

- But time intervals: $\delta[dt] = dt[t+\delta t] - dt[t]$
$$= +dt\left(\frac{\dot a}{a}\right)\delta t$$
along light cone

• Solid angles: From Lorentz transformation:

$$\delta[d\Omega] = d\Omega[t+\delta t] - d\Omega(t) = +2\,d\Omega\,\frac{\dot a}{a}\,\delta t$$

Net effect of expansion on intensity

$$I_\nu[t+\delta t, \nu+\delta\nu] = dN\left(\frac{h\nu}{dA_\perp\, h\nu}\right) \times \frac{1}{dt(t+\delta t)\, d\Omega(t+\delta t)}$$

$$= dN\left(\frac{h\nu}{dA_\perp\, h\nu}\right) \frac{1}{dt\left(1+\frac{\dot a}{a}\delta t\right) d\Omega\left(1+\frac{2\dot a}{a}\delta t\right)}$$

$$\therefore\ I_\nu[t+\delta t, \nu+\delta\nu] = I_\nu(t,\nu)\left[1 - \frac{3\dot a}{a}\delta t\right]$$

Taylor Expand:

$$\cancel{I_\nu} + \frac{\partial I_\nu}{\partial t}\delta t + \frac{\partial I_\nu}{\partial \nu}\delta\nu = \cancel{I_\nu} - \left(\frac{3\dot a}{a}\delta t\right) I_\nu$$

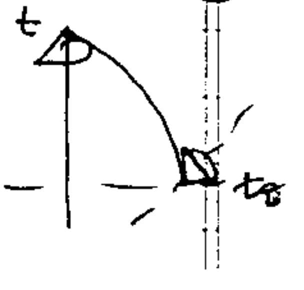
Divide by δt

$$\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial \nu}\frac{\delta\nu}{\delta t} = -\left(\frac{3\dot a}{a}\right) I_\nu$$

$$\boxed{\frac{dI_\nu}{dt} = -\frac{3\dot a}{a} I_\nu \quad (1)}$$

$\therefore\ \dot a > 0 \Rightarrow$ I_ν decreases due to expansion

Integrate eq. (1) from $t_E \to t_0$



$$\int_{I_{\nu_E}(t_E)}^{I_\nu(t)} \frac{dI_\nu}{I_\nu} = -3\int_{t_E}^{t} \frac{\dot a}{a}\, dt$$

$$\ln\left[\frac{I_\nu(t)}{I_{\nu_E}(t_E)}\right] = -3\ln\left(\frac{a(t)}{a(t_E)}\right) = -\ln\left[\frac{a(t)}{a(t_E)}\right]^3$$

$$\Rightarrow \quad I_\nu(t) = I_{\nu_E}(t_E)\left(\frac{a(t_E)}{a(t)}\right)^3$$