$$(\nu_{obs})_{21} = \frac{(\nu_E)_{21}}{1+z_{opt}} = \frac{(\nu_E)_{21}}{(\nu_E)_{opt}} \times (\nu_{obs})_{opt}$$

$\Rightarrow$ Ratio of frequencies not altered by cosmic expansion since

$$\left(\frac{\nu_{21}}{\nu_{opt}}\right)_{obs} = \left(\frac{\nu_{21}}{\nu_{opt}}\right)_{intrinsic}$$

Furthermore, these frequency ratios depend on physical constants: $\alpha^2 g_p m_e/m_p$ - Demonstration that dimensionless physical constants haven't varied!

Expansion Dynamics

How does $a(t)$ behave as a function of time? To answer this question we must insert FRW metric into the Einstein field equations and then solve for $a(t)$. Also impose homogeneous & isotropic symmetry!

Einstein field eqs.

$$R_{ab} - \frac{1}{2} g_{ab} R^c_c = -\kappa T_{ab}$$

$$\kappa = \frac{8\pi G}{c^4}$$

Terms: R_{ab} is the Ricci tensor

$$R_{ab} = \Gamma^c_{ac,b} - \Gamma^c_{ab,c} + \Gamma^c_{ad}\Gamma^d_{cb} - \Gamma^c_{ab}\Gamma^d_{cd}$$

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\left[g_{da,b} + g_{db,a} - g_{ab,d}\right]$$

Looks worse than it is. Imposed symmetry reduces no. of independent components.

Stress Energy Tensor:

Perfect fluid: no viscosity, no conduction, no microscopic transpo[rt] proce[sses]

$$T^{ab} = \left(\rho + \frac{P}{c^2}\right) u^a u^b - P g^{ab} \quad : ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r\ldots\right.$$

Because of symmetry $\rho = \rho(t)$; $P = P(t)$.

much work

Friedmann/

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) a \qquad (1)$$

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - c^2 k \qquad (2)$$

$T^{ab}{}_{;b} = 0$ follows from Einstein eqs.

$$T^{ab}{}_{;b} = T^{ab}{}_{,b} + \Gamma^a_{db} T^{db} + \Gamma^b_{db} T^{ad}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \left(\rho + \frac{P}{c^2}\right)\frac{3\dot{a}}{a} = 0} \qquad (3)$$

Equation of state

$$P = w\rho c^2 \quad w = \text{const}.$$

$$\frac{d\rho}{dt} + (\rho + w\rho)\frac{3\dot a}{a} = 0$$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -3(1+w)\frac{\dot a}{a}$$

$$\frac{d\ln\rho}{dt} = -3(1+w)\frac{d\ln a}{dt}$$

$$\Rightarrow \quad \ln\rho = -3(1+w)\ln a + \text{const}$$

$$\boxed{\rho = \text{const}\; a^{-3(1+w)}}$$

(1) ~~expans~~ matter dominated Universe: $w=0$

$$\rho \propto a^{-3}$$

(2) Radiation-pressure: early universe: $w = 1/3$

$$\rho \propto a^{-4}$$

(3) Vacuum Energy: $w = -1$

$$\rho = \text{const}$$

Let's look at solutions

① Simple Case: Spatially flat, $K=0$
non-relativistic matter, $w=0$, $\rho=\rho_0\left(\frac{a_0}{a}\right)^3$

From eq. (2) we have:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2$$

at current epoch $\left(\frac{\dot{a}}{a}\right)_0^2 = \frac{8\pi G}{3}\rho_0$

Since current Hubble constant, $H_0 = \left(\frac{\dot{a}}{a}\right)_0$
this defines the critical density

$$\boxed{\rho_0 = \rho_{crit} = \frac{3H_0^2}{8\pi G} = 1.88\times10^{-29}h^2\,g\,cm^{-3}}$$

this defines a critical density because in simple models in which $\rho > \rho_0$, $K=+1$
$\rho < \rho_0$, $K=-1$

Evolution: $a\dot{a}^2 = \frac{8\pi G}{3}\rho a^3 = \frac{8\pi G}{3}\rho_0 a_0^3$

$$a^{1/2}\dot{a} = \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3}$$

Integrate: $\int_0^a a^{1/2}da = \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3}\int_0^t dt$

$$\frac{2}{3}a^{3/2} = \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3}\;t$$

$$\frac{2}{3}[a(t)]^{3/2} = \sqrt{\frac{8\pi G}{3} a_0^3 \cdot \frac{3H_0^2}{8\pi G}}\; t = \sqrt{a_0^3 H_0^2}\; t$$

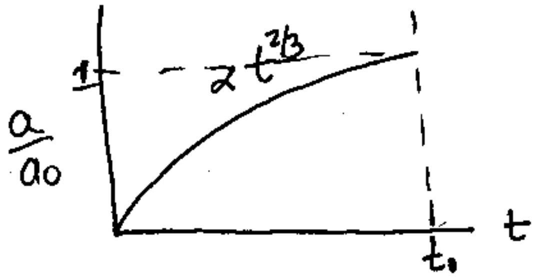
Therefore: $\boxed{\frac{2}{3} a^{3/2} = a_0^{3/2} H_0 t \qquad (4)}$

Normalize at present: $\boxed{\frac{2}{3} a_0^{3/2} = a_0^{3/2} H_0 t_0 \qquad (5)}$

$\Rightarrow \qquad t_0 = \frac{2}{3} H_0^{-1}$: age $= \frac{2}{3} \times$ Hubble time

$$t_0 = \frac{2}{3} \times 10.3\, h^{-1}\ \text{Gyr} = 6.9\, h^{-1}\ \text{Gyr}$$

For $h \simeq 0.7 \Rightarrow t_0 \simeq 10$ Gyr (too young!)



Universe decelerates

Hubble Parameter : Divide (4) by (5)

$$\left(\frac{a}{a_0}\right)^{3/2} = \left(\frac{t}{t_0}\right) \Rightarrow a = a_0 \left(\frac{t}{t_0}\right)^{2/3}$$

or $\quad a = (\text{const})\, t^{2/3}$

$\quad\ \ \dot{a} = \frac{2}{3} (\text{const})\, t^{-1/3}$

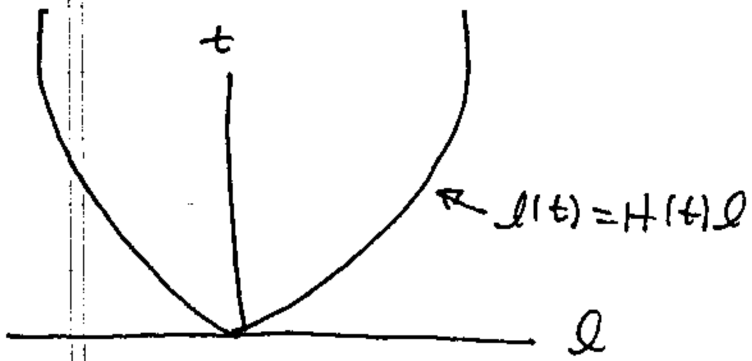
$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t}$$

as $t \to \infty$, $H(t) \to 0$

Thus, recession of comoving (FO) galaxies at time t:

$$u(t) = H(t)\, l$$

$$\therefore \lim_{t \to 0} u(t) = 0$$



galaxies stop receding as $t \to \infty$

Back to Friedmann equation

(A) $\boxed{\text{General Considerations}}$ Go back to 2nd eq. (2)

$$\dot{a}^2 - \frac{8\pi G \rho}{3} a^2 = -c^2 K$$

Evaluate this at present epoch:

$$(\dot{a})_0^2 - \frac{8\pi G \rho_0}{3} a_0^2 = -c^2 K$$

Define total Density parameter, Ω_T:

$$\rho_0 = \Omega_T \cdot \rho_{crit} = \Omega_T \cdot 3H_0^2 / 8\pi G$$

Therefore $$(\dot{a})_0^2 - \frac{8\pi G}{3}\left(\Omega_T \frac{3H_0^2}{8\pi G}\right) a_0^2 = -c^2 K$$

Divide by $H_0^2 a_0^2 \Rightarrow$

$$\boxed{1 - \Omega_T = -\frac{c^2 K}{H_0^2 a_0^2} \qquad (6)}$$

$$\Omega_T - 1 = \frac{c^2 K}{(H_0 a_0)^2}$$

Thus, without even knowing ~~what~~ the constituents of the universe we see that if

$$\boxed{\begin{array}{l} \Omega_T > 1 \ (\rho_0 > \rho_{crit}), \text{ then } K = +1 \\ \Omega_T < 1 \ (\rho_0 < \rho_{crit}), \text{ then } K = -1 \\ \Omega_T = 1 \ (\rho_0 = \rho_{crit}), \text{ then } K = 0 \end{array}} \quad (I)$$

(B) Back to Friedmann eq. (1)

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a$$

Suppose $\rho + \frac{3P}{c^2} > 0$

(as would be the case for mixture of matter and radiation). Then the above equation shows that

$$\boxed{\frac{\ddot{a}}{a} < 0}$$

Thus in case of matter and/or radiation alone the cosmic expansion slows down.
Consider non-relativistic matter alone.

From $\rho \propto a^{-3(1+w)}$
Since $w = 0$

$$\rho_0 = \rho_0 (a_0/a)^3$$

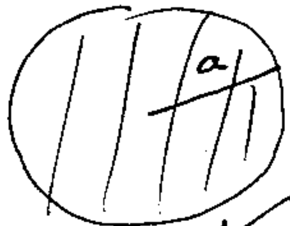
From second Friedmann equation:

$$\frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3}\rho a^2 = -\frac{c^2K}{2}$$

$$\frac{1}{2}\dot{a}^2 - \frac{4\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 a^2 = -\frac{c^2K}{2}$$

$$\frac{1}{2}\dot{a}^2 - \left(\frac{4\pi G}{3}\rho_0 a_0^3\right)\Big/ a = -\frac{c^2K}{2}$$

Mass contained in uniform sphere with radius a, density ρ stays constant

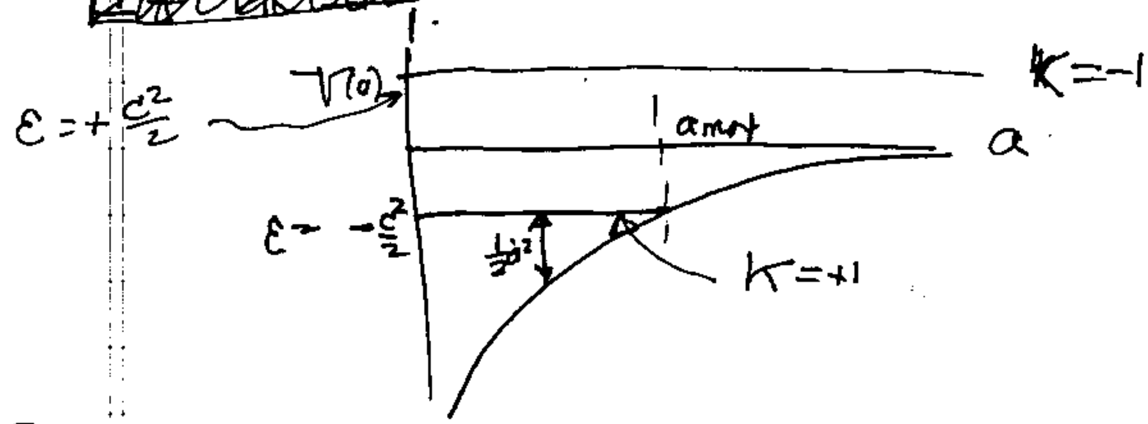


$M(a_0) = \frac{4\pi G}{3}\rho_0 a_0^3$

$$\frac{1}{2}\dot{a}^2 - \underbrace{\frac{GM(a_0)}{a}}_{V(a)} = -\frac{c^2K}{2}$$

unit mass Particle in a "1/r" potential well with V(a) potential

~~[illegible]~~



K=+1

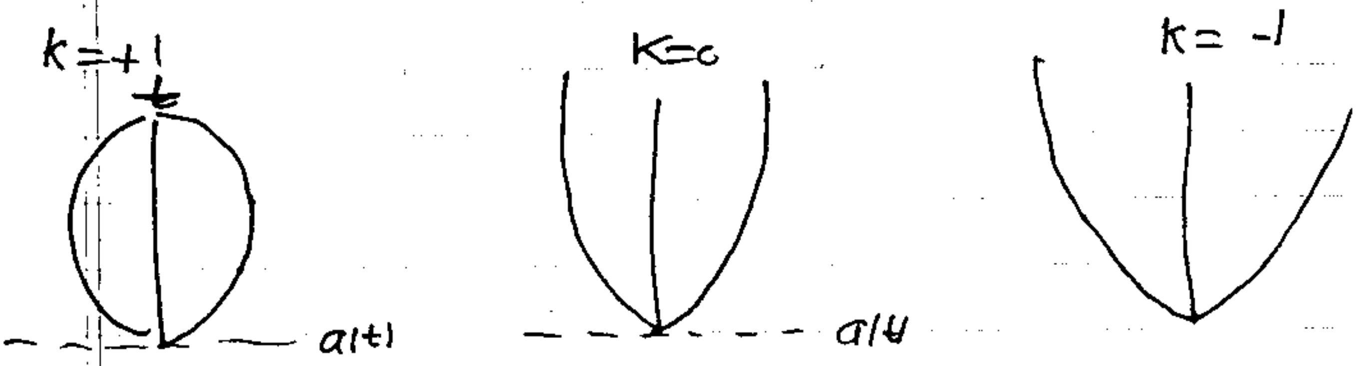
Particle decelerates, $(1/2)\dot{a}^2$ decreases until it = 0 at a_{max}. Then since deceleration continues, Universe contracts until $a=0$ again.

K=−1

On the other hand if K=−1, the universe expands ~~[illegible]~~ forever, even though it decelerates

$K=0$ is special, critical case of expansion forever, but reaching $a \to \infty$ with $\dot{a}=0$

~~Sce~~ Schematically we have:



Significance of Einstein-deSitter ($K=0$, $\rho_0=\rho_{crit}$) Universe

Back again to 2nd Friedmann Equation

$$\dot{a}^2 - \frac{8\pi G}{3}(\rho a^2) = -c^2 K$$

Notice that in case of ρ being dominated by radiation we saw $\rho \propto a^{-4}$.

Therefore $\rho a^2 \propto a^{-2}$ Radiation

$\rho a^2 \propto a^{-1}$ Matter

Therefore at early times the magnitude of the "potential" term dominates the curvature (or energy term) and to an excellent approximation

$$\left(\frac{\dot{a}}{a}\right)^2 \to \frac{8\pi G}{3}\rho$$

Thus at early times $\rho = \dfrac{3H^2(t)}{8\pi G}$

density equals the critical density to high prec.
But as we shall see, the current energy density of the current Universe is a substantial fraction of the critical density.

Why after billions of years is it that ρ is not very different from ρ_{crit}?

Other special cases

(B) Relativistic matter, i.e., radiation dominated.
When $K = 0$

$$\dot{a}^2 = \frac{8\pi G}{3}(\rho a^2)$$

Since $\rho a^2 \propto a^{-2}$ in this case

$$(a\dot{a})^2 = \text{const}$$

$$a\frac{da}{dt} = \text{const} \Rightarrow a^2 \propto t$$

$$a \propto t^{1/2}$$ (expands slower than for matter alone)

(C) Vacuum Energy Dominated:
$w = -1$, $\rho = \rho_v$ (no time dependence)
For $K = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi \rho_v}{3} = H^2 = \text{const}$$

$$\therefore \ \frac{1}{a}\frac{da}{dt} = H \implies \frac{da}{a} = \cancel{Ht}$$

Integrate. $\ln(a) = Ht + \text{const}$

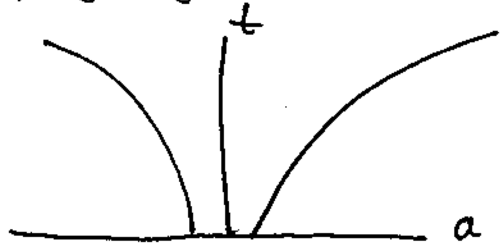
$$a(t) \propto e^{Ht}; \quad H = \sqrt{\frac{8\pi G\rho_v}{3}}$$

This is the famous deSitter Universe:

<u>Properties</u>

(1) It <u>accelerates</u>: $\frac{\ddot{a}}{a} = H^2$

Implying $\ddot{a} > 0$



(2) Devoid of matter since $\rho = \rho_v$

(3) $a(t)$ does not $\to 0$ at early times!

As I will show, this is not a bad approximation for behavior of the current Universe, which, as we shall see, is <u>accelerating</u>. While current ρ is dominated by vacuum energy density, this was not the case in the past when matter and radiation were dominant

Aside on cosmological constant

Einstein modified field eqs. as follows:

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = -K T'_{ab}$$

where T'_{ab} is stress-energy tensor <u>without</u> vacuum contribution.

Rewrite: $R_{ab} - \frac{1}{2} g_{ab} R = -K T'_{ab} - \Lambda g_{ab}$

Original field eqs: $R_{ab} - \frac{1}{2} g_{ab} R = -K T_{ab}$

Therefore $\Lambda g_{ab} = K (T_{ab} - T'_{ab})$

$$\text{or } \Lambda g_{ab} = K \left[\left(\rho_{vac} + \frac{P_{vac}}{c^2} \right) u_a u_b - P_{vac} g_{ab} \right]$$

But since $P_{vac} = -\rho_{vac} c^2$

$$\Lambda g_{ab} = -K(-\rho_v c^2) g_{ab} = K \rho_v c^2 g_{ab}$$

$$\text{or } \Lambda = \frac{8\pi G}{c^4} \rho_{vac} c^2$$

$$\boxed{\Lambda = \frac{8\pi G}{c^2} \rho_{vac}}$$

Most General Case

$$\dot{a}^2 - \frac{8\pi G}{3}\rho a^2 = -c^2 K$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{c^2 K}{a^2} \quad (8)}$$

Divide by H_0^2 : $\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3H_0^2} - \frac{c^2 K}{a^2 H_0^2}$

or $$\boxed{\frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho(t)}{\rho_{crit}} - \frac{c^2 K}{a^2 H_0^2} \quad (9)}$$

$\rho_{crit} = 3H_0^2/8\pi G$ = current critical density

relativistic matter

Let $\rho(t) = \rho_v + \rho_M(t) + \rho_R(t)$

Recall: $\rho_M(t) = \rho_{M0}\left(\frac{a_0}{a}\right)^3$

$$\rho_R(t) = \rho_{R0}\left(\frac{a_0}{a}\right)^4$$

ρ_v = const.

$$\rho(t) = \rho_v + \rho_{M0}\left(\frac{a_0}{a}\right)^3 + \rho_{R0}\left(\frac{a_0}{a}\right)^4$$

Define density parameters as follows

$$\rho_v = \frac{3H_0^2}{8\pi G}\Omega_\Lambda \,; \quad \rho_{M0} = \frac{3H_0^2}{8\pi G}\Omega_M \,; \quad \rho_{R0} = \frac{3H_0^2}{8\pi G}\Omega_R$$

$$\therefore \rho(t) = \frac{3H_0^2}{8\pi G}\left[\Omega_\Lambda + \Omega_M\left(\frac{a_0}{a}\right)^3 + \Omega_R\left(\frac{a_0}{a}\right)^4\right]$$

at $t = t_0$

$$\rho_0 = \rho_{crit}\,(\Omega_\Lambda + \Omega_M + \Omega_R)$$

From Friedmann eq. (8) we have when $t = t_0$

$$H_0^2 = \frac{8\pi G \rho_0}{3} - \frac{c^2 K}{a_0^2}$$

Divide by H_0^2:

$$1 = \frac{\rho_0}{\rho_{crit}} - \frac{c^2 K}{(a_0 H_0)^2}$$

solve for current radius

$$\frac{K}{(a_0/R_H)^2} = \frac{\rho_0}{\rho_{crit}} - 1 = \Omega_{TO} - 1$$

$$a_0 = R_H \sqrt{\frac{K}{\Omega_{T}-1}}$$

$R_H = \frac{c}{H_0}$

Since $\rho_0 = \rho_{vac} + \rho_M(0) + \rho_R(0)$, we have

$$\boxed{1 = \Omega_\Lambda + \Omega_m + \Omega_R + \Omega_K,} \quad (10)$$

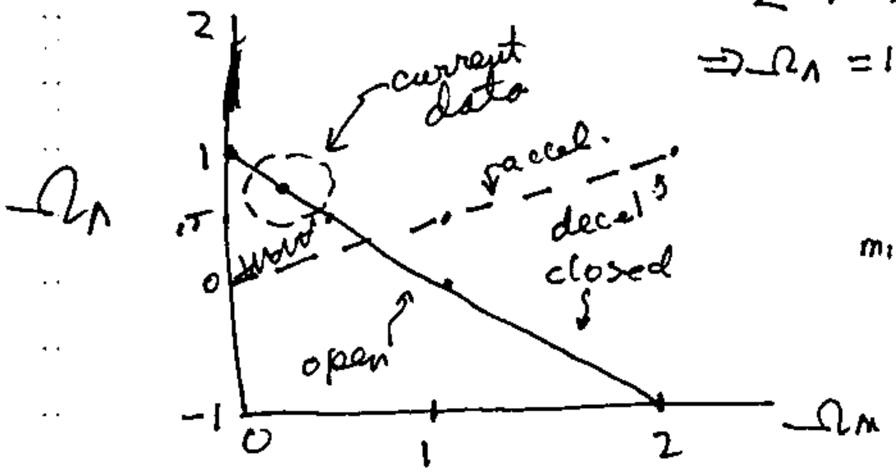
where $\Omega_K \equiv \dfrac{-c^2 K}{(a_0 H_0)^2}$

Rough estimates $\Omega_\Lambda, \Omega_m \sim 1$: $\Omega_R = \dfrac{aT^4}{\rho_{crit} c^2} = \dfrac{7.57\times 10^{-15}(2.72)^4}{1.88\times 10^{-29}(3\times 10^{10})^2 h^2}$

$\Omega = 2.44\times 10^{-5} h^{-2}$: Ignore it

To a good approx: $\Omega_\Lambda + \Omega_m = 1 + \dfrac{c^2 K}{(a_0 H_0^2)}$

- Geometry: If $\Omega_\Lambda + \Omega_m > 1 \Rightarrow K = +1$
  $< 1 \Rightarrow K = -1$



$\Rightarrow \Omega_\Lambda = 1 - \Omega_m$: boundary line

- As Ω_Λ moves minimum Ω_m to close the Universe must increase

- **Decelerating or Accelerating**

Back to 1st Friedmann equation:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)a$$

Since $P = (1+w)\rho c^2$ we have

$$\ddot{a} = -\frac{4\pi G}{3}\rho(1+3w)a$$

Mult. by $\frac{a}{\dot{a}^2}$

(Dimensionless) $$\frac{a\ddot{a}}{a^2} = -\frac{4\pi G\rho}{3}(1+3w)\left(\frac{a}{\dot{a}}\right)^2$$

or $$\frac{a\ddot{a}}{a^2} = -\frac{4\pi G\rho}{3H^2}(1+3w)$$

More accurately.. $$\frac{a\ddot{a}}{a^2} = -\frac{4\pi G}{3H^2}\sum_i^{N_{comp}}\rho_i(1+3w_i)$$

$$\left(\frac{a\ddot{a}}{a^2}\right)_0 = -\frac{4\pi G}{3H^2}\Big\{\underset{(w=0)}{\rho_M} + \underset{(w=1/3)}{\rho_R\, 2} - \underset{w=-1}{2\rho_{vac}}\Big\}_0$$

Evaluate now and let $q_0 \equiv -\frac{a\ddot{a}}{a^2}$ be deceleration parameter

$$q_0 = \frac{1}{2\left(\frac{3H_0^2}{8\pi G}\right)}\left[\rho_M + 2\rho_R - 2\rho_{vac}\right]$$

$$q_0 = \frac{1}{2}\left[\Omega_M + 2\Omega_R - 2\Omega_\Lambda\right]$$

Since $\Omega_R \ll \Omega_m$ or Ω_{vac} $\Rightarrow q_0 \cong \frac{1}{2}(\Omega_m - 2\Omega_\Lambda)$

Deceleration $\Rightarrow q_0 > 0 \Rightarrow$ boundary $\Omega_m > 2\Omega_\Lambda$
(back to Figure)

$\Omega_\Lambda = \frac{1}{2}\Omega_m$ boundary!

Figure implies:

- Current data (which we will discuss) implies
  - Universe is spatially flat since $\Omega_\Lambda + \Omega \approx 1$
  - " is accelerating (For $\Omega_m \approx .3$, $\Omega_\Lambda < .15$ for deceleration)
- Even for significant vacuum energy, which acts like "negative mass" that induces acceleration, with sufficient Ω_m, universe can still decelerate
- Likewise: sufficiently large Ω_Λ can always give rise to acceleration, even for high Ω_M

General solutions for scale factor $a(t)$ and age

Back to eq. (9)

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{c^2 K}{a^2}$$

Divide by H_0^2

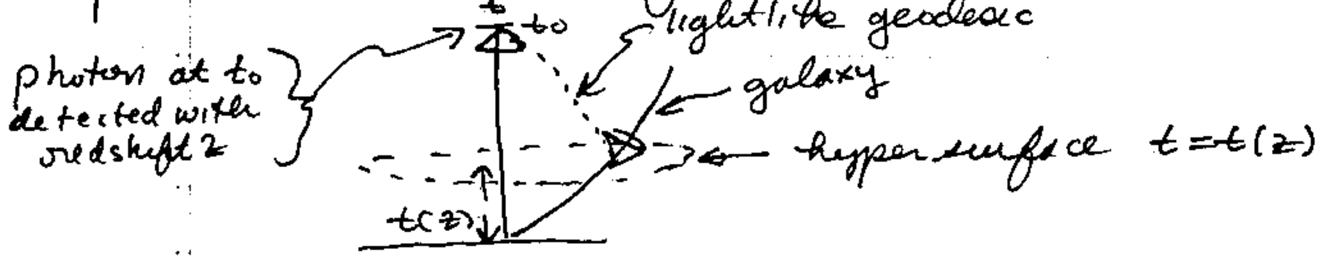
$$\frac{1}{H_0^2}\left(\frac{\dot a}{a}\right)^2 = \frac{\rho}{3H_0^2/8\pi G} - \frac{c^2K}{a^2H_0^2} = \frac{\sum_i \rho_i}{\rho_{crit}} - \frac{c^2K}{a^2H_0^2}$$

$$\frac{1}{H_0^2}\left(\frac{\dot a}{a}\right)^2 = \Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_R\left(\frac{a_0}{a}\right)^4 - \frac{c^2K}{\left(\frac{a}{a_0}\right)^2(a_0H_0)^2}$$

$$\frac{1}{H_0^2}\left(\frac{\dot a}{a}\right)^2 = \Omega_\Lambda + \Omega_m\left(\frac{a_0}{a}\right)^3 + \Omega_R\left(\frac{a_0}{a}\right)^4 + \Omega_K\left(\frac{a_0}{a}\right)^2$$

First, let's compute age of the Universe corresponding to redshift z.

That is cosmic time since $t=0$ at which photon with redshift z was emitted.



$$\frac{1}{H_0}\left(\frac{1}{a}\frac{da}{dt}\right) = \sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_R(1+z)^4 + \Omega_K(1+z)^2}$$

Since $a = a_0(1+z)^{-1} \Rightarrow da = -a_0(1+z)^{-2}dz$

$$\frac{da}{a} = -\frac{a_0(1+z)^{-2}}{a_0(1+z)^{-1}}dz = -\frac{dz}{(1+z)}$$

Therefore: $$dt = -\frac{dz}{H_0(1+z)\sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_R(1+z)^4 + \Omega_K(1+z)^2}}$$

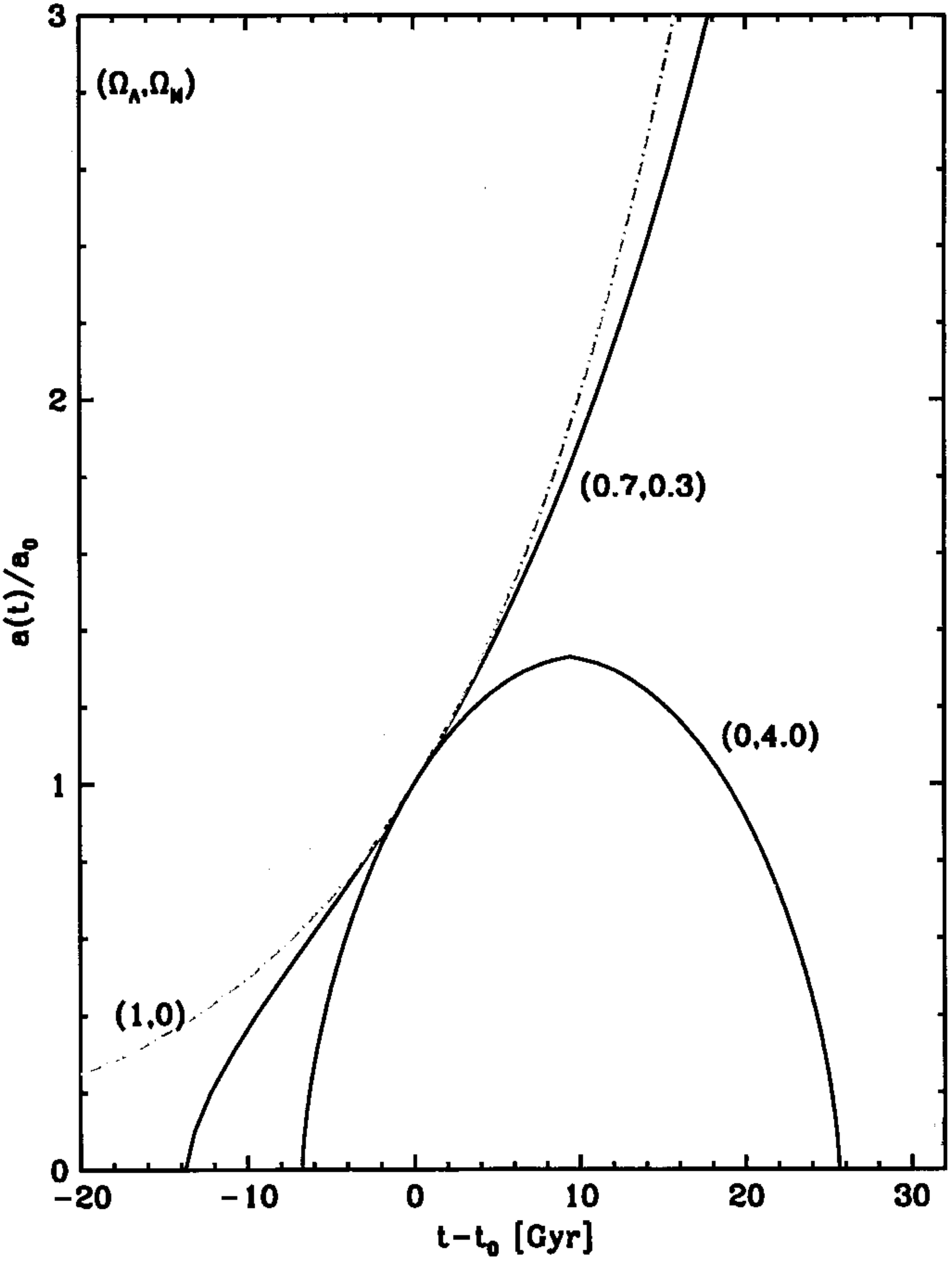
Let $x \equiv a/a_0 = (1+z)^{-1} \Rightarrow \frac{dz}{(1+z)} = -\frac{dx}{x}$

Assume $t=0$ corresponds to $z \to \infty$

or $t=0$ corresponds to $x=0$

$$\boxed{t(z) = \frac{1}{H_0}\int_0^{\frac{1}{1+z}} \frac{dx}{x\sqrt{\Omega_\Lambda + \Omega_K x^{-2} + \Omega_m x^{-3} + \Omega_R x^{-4}}}} \quad (1)$$

Age of the Universe obtained by setting $z=0$



$(\Omega_\Lambda, \Omega_M)$
(0.7,0.3)
(0,4.0)
(1,0)
$a(t)/a_0$
$t-t_0$ [Gyr]
0
1
2
3
−20
−10
0
10
20
30

Figure shows plot of $a(t)/a_0$ versus $t - t_0$; i.e., time relative to present epoch

Green Curve: de Sitter Universe

$\Omega_\Lambda = 1$, $\Omega_M = 0$, $\Omega_K = 0$, $h = 0.7$, $\Omega_T = 1$, $\Omega_R =$

Universe accelerates at all epochs.

- shows effects of vacuum energy
- where we are headed in the future

Blue Curve: Best estimate of our Universe

$\Omega_\Lambda = 0.7$, $\Omega_M = 0.3 - \Omega_R$, $\Omega_R = \frac{2.44 \times 10^{-5}}{h^2}$, $h = .7$

$\Omega_T = 1$

- Introduction of Ω_M and Ω_R causes deceleration at earlier epochs, not that far back in time
- Switchover to acceleration rather recent

Red Curve:

$\Omega_\Lambda = 0$, $\Omega_M = 4$, $\Omega_R = 2.44 \times 10^{-5}/h^2$

- Universe decelerates at all epochs
- Has maximum radius $a = a_{max}$ at which $\dot{a}/a = 0$
- Recollapse in future

Flatness $\longrightarrow$

Flatness problem

Back to Friedmann eq.

$$\dot{a}^2 = \frac{8\pi G\rho}{3} a^2 - c^2 K$$

Divide by a^2

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{c^2 K}{a^2}$$

Divide by $\left(\frac{\dot{a}}{a}\right)^2 = H^2(t)$

$$1 = \frac{8\pi G\rho}{3H^2} - \frac{c^2 K^2}{H^2(t) a^2}$$

Define: $\Omega(t) \equiv \frac{\rho(t)}{\rho_{crit}(t)} = \frac{\rho}{3H^2(t)/8\pi G}$ { total density parameter at time t

therefore $1 = \Omega(t) - \frac{c^2 K}{H^2(t) a^2}$ or $\boxed{\Omega(t) - 1 = \frac{c^2 K}{H^2(t) a^2}}$ (12)

Evaluate eq. (a) at present time $t = t_0$

$$\Omega - 1 = \frac{c^2 K}{H_0^2 a_0^2}$$: As a result: compare with (12)

$$H_0^2 a_0^2 (\Omega - 1) = H^2(t) a^2(t) (\Omega(t) - 1)$$

or $$\boxed{\Omega(t) - 1 = \frac{H_0^2}{H^2(t)} \frac{a_0^2}{a^2} (\Omega - 1)}$$ (13)

Assume t is so small that we are in radiation-dominated era! $\rho_R \gg \rho_m$ or ρ_{vac} at early times:

Since curvature terms are negligible back then,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\underset{\text{mass density}}{\rho(t)} = \frac{8\pi G}{3}\left(\frac{\underset{\text{energy density}}{u_r}}{c^2}\right)$$

$$\text{or} \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\hat{a}_0 T^4(t)}{c^2}\right) \quad \text{black body}$$

I am going to assume without proof (I'll do this later in CMB discussion) that $T(t) \propto \frac{1}{a(t)}$

$$T(z) = T(0)(1+z), \quad \text{where } T(0) = 2.72\,K$$

$$\therefore \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \times \left(\frac{7.57\times10^{-15} \times (2.72)^4}{(3\times10^{10})^2}\right)(1+z)^4$$

$$\text{or } H^2(t) = 2.58\times10^{-40}\,(1+z)^4$$

Since $a_0/a = 1+z$; $H_0 = 3.25\times10^{-18}\,h\ s^{-1}$ we have from eq. (13)

$$\Omega(t) - 1 = \frac{(3.25\times10^{-18})^2 h^2}{2.58\times10^{-40}}\,\frac{(1+z)^2}{(1+z)^4}(\Omega - 1)$$

$$\text{or } \Omega(t) - 1 = \frac{4.07\times10^4\, h^2 (\Omega-1)}{(1+z)^2}$$

The earliest epoch at which we could rationally evaluate $\Omega(t)$ would be the Planck time

As we shall see, this is epoch at which $T_p \simeq 10^{32} K$ or at which $1+z = \frac{T_P}{T_0} = 3\times10^{31}$

$$\therefore \quad \Omega(t_{planck}) - 1 \simeq \frac{4.1\times10^4 h^2}{(3\times10^{31})^2}(\Omega - 1)$$

$$\boxed{\Omega(t_{planck}) - 1 \simeq 3\times10^{-59} h^2 (\Omega - 1)}$$

$h = .7 \qquad \Omega(t_{planck}) - 1 = 1.5\times10^{-59}(\Omega - 1)$

Therefore: until very recently Ω was uncertain but best bets were

$$0.1 \leq \Omega \leq 2$$

Thus, if Ω today is not exactly equal to one, we require incredibly fine adjustment of "initial" total density parameter

$$1 - 1.29\times10^{-59} \leq \Omega(t_{planck}) \leq 1 + 1.47\times10^{-59}$$

$$0.9999999\ldots 971 \leq \Omega(t_{planck}) \leq 1.0000 \ldots\ldots 147$$

(A) Incredibly fine tuning of initial Universe to get $\Omega = O(1)$ but not precisely $= 1$.

(B) Price to be paid: If $\Omega(t_{planck}) = 1.1$, at Planck time ($\simeq 10^{-43}$ sec), Universe would recollapse in 10^{-42} sec