→ Spiral Wave Dynamics

Useful to develop transport equation for spiral wave intensity.

∴ → Variational theory { Whitham; R. L. Dewar, Ap. J. 174, 30 1972 }

→ action / energy } density equations

Now, consider action integral for wave system (i.e. linear, so $\mathcal{L} \sim a^2$)

$$S = \int d^3x \int dt \; \mathcal{L}$$

where $\mathcal{L} \sim O(a^2)$ → linear wave

and eqn. of motion: $\delta S = 0$.

Now, assume < wave packet / wave train > solution

so if

$$\mathcal{L} = \mathcal{L}(\phi)$$

i.e. $$\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - T\,\frac{(\nabla \phi)^2}{2}$$

eikonal phase

then $\phi = a\, e^{i\Theta}$

amplitude

$\therefore\ \mathcal{L} = \left(\frac{\rho}{2}\omega^2 - T k^2\right) a^2 \qquad \omega = -\partial_t \Theta,\quad \underline{k} = \underline{\nabla}\Theta$

$= D(k,\omega)\, a^2$

dispersion fctn.

Now, $\delta S = 0 \Rightarrow \delta S/\delta a = 0$

$\delta S/\delta\Theta = 0$

$\delta S/\delta a = 0 \Rightarrow D(k,\omega) = 0$

i.e. dispersion relation

$\delta S/\delta\Theta = 0 \Rightarrow$ phase symmetry

i.e. action conservation results from required phase symmetry on wave train $(\Theta \to \Theta + \delta\Theta)$

$$\delta S/\delta\Theta = \int d^3x \int dt \left[\frac{\partial \mathcal{L}}{\partial \Theta_t}\cdot\delta\Theta_t + \frac{\partial \mathcal{L}}{\partial \underline{\nabla}\Theta}\cdot\delta\underline{\nabla}\Theta + \frac{\partial \mathcal{L}}{\partial \Theta}\delta\Theta\right]$$

$$= \iint \left[-\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \Theta_t} - \frac{\nabla\cdot\partial \mathcal{L}}{\partial \underline{\Theta_x}} + \frac{\partial \mathcal{L}}{\partial \Theta}\right]\delta\Theta$$

$\partial S / \partial \theta = 0$

$$0 = \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \omega} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial \underline{k}} + \frac{\partial \mathcal{L}}{\partial \theta}$$

$$= \frac{\partial}{\partial t}\left(\partial D/\partial \omega \, a^2 \right) - \nabla \cdot \left((\partial D/\partial \underline{k})\, a^2 \right)$$

now $\quad \mathcal{L} = \left(\frac{\rho \omega^2}{2} - \frac{T k^2}{2} \right) a^2$

$$\frac{\partial \mathcal{L}}{\partial \omega} = (\rho \omega)\, a^2$$

$$= \frac{\rho \omega^2}{\omega} a^2$$

$$= \frac{a^2}{\omega} \left(\frac{\rho \omega^2}{2} + \frac{T k^2}{2} \right)$$

wave energy $\qquad \omega^2 = \frac{T}{\rho} k^2$

$$= \mathcal{E}/\omega$$

so $\frac{\partial \mathcal{L}}{\partial \omega} \equiv$ wave action density

($\mathcal{L} * t \to$ action / volume)

so $\quad \frac{\partial D}{\partial \omega} a^2 \equiv N$

$$\boxed{\begin{aligned} &\frac{\partial N}{\partial t} + \underline{\nabla}\cdot(\underline{V}_{gr} N) = 0 \\ &\underline{V}_{gr} = -\frac{\partial D/\partial \underline{k}}{\partial D/\partial \omega} \end{aligned}}$$

→ action density continuity

→ group velocity

Now, $\quad N = \mathcal{E}/\omega$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\left(\mathcal{E}/\omega\right) + \nabla\cdot\left(V_{gr}\frac{\mathcal{E}}{\omega}\right) = 0}$$

$$\frac{1}{\omega}\frac{\partial \mathcal{E}}{\partial t} - \frac{1}{\omega^2}\frac{\partial \omega}{\partial t}\mathcal{E} + \frac{1}{\omega}\nabla\cdot(V_{gr}\mathcal{E}) - \mathcal{E}\frac{\underline{V}_{gr}\cdot\nabla\omega}{\omega^2} = 0$$

$$\frac{1}{\omega}\left(\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla}\cdot(V_{gr}\mathcal{E})\right) - \frac{\mathcal{E}}{\omega^2}\left(\frac{d\omega}{dt}\right) = 0$$

$\underline{\text{if}}\quad d\omega/dt = 0 \quad \Rightarrow \quad \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial \underline{k}}\cdot\frac{d\underline{k}}{dt} + \frac{\partial \omega}{\partial \underline{x}}\cdot\frac{d\underline{x}}{dt} = 0$

eikonal eqns

$$\Rightarrow \frac{\partial \omega}{\partial t} = 0$$

i.e. if no explicit time dependence in frequency

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(\underline{v}_{gr} \mathcal{E} \right) = 0$$

→ Form of Action?

→ plasma (e.s.) $\nabla^2 \phi = -4\pi \rho$

$$\mathcal{L} = -\rho\phi + \frac{(\nabla\phi)^2}{8\pi}$$

i.e. $\frac{\delta S}{\delta\phi} = \iint \left[-\frac{\rho}{2} \delta\phi + \frac{\nabla\phi}{4\pi} \delta\nabla\phi \right]$

$= \iint \left[-\frac{\rho}{2} - \frac{\nabla^2\phi}{4\pi} \right] \delta\phi$

$$\rho_{k,\omega} = \chi_{k,\omega} \phi_{k,\omega}$$

$$\mathcal{L}(k,\omega) = \left[-\chi_{k,\omega} + \frac{k^2}{4\pi} \right] \phi_{k,\omega}^2$$

$$= \left(1 - \frac{4\pi \chi_{k,\omega}}{k^2} \right) \frac{k^2 \phi_{k,\omega}^2}{8\pi}$$

$$= \epsilon(k,\omega) \frac{k^2 \phi_{k,\omega}^2}{8\pi}$$

$$N = (\partial\epsilon/\partial\omega) \frac{k^2 \phi_{k,\omega}}{8\pi}$$

$$= (\partial\epsilon/\partial\omega) \frac{\langle (\nabla\phi)^2 \rangle}{8\pi}$$

GK

$$\hat{\phi} = \frac{|e|\tilde{\phi}}{T}$$

$$\frac{\hat{n}_0}{n_0}\bigg|_{GC} = \hat{\phi} - \rho_s^2 \nabla^2 \hat{\phi}$$

$$\mathcal{L} = \left(\frac{\hat{n}_i}{n_0} - \frac{\hat{\phi}}{2}\right)\hat{\phi} - \rho_s^2 \frac{(\nabla\hat{\phi})^2}{2}$$

$$\delta \iint \mathcal{L} = \iint \left\{ \left(\frac{\hat{n}_i}{n} - \hat{\phi}\right) \delta\hat{\phi} + \left(\rho_s^2 \nabla^2 \hat{\phi}\right)\delta\hat{\phi} \right\} = 0$$

$$\frac{\hat{n}_i}{n_0} - \hat{\phi} + \rho_s^2 \nabla^2 \hat{\phi} = 0$$

$$\frac{\hat{n}_i}{n} = \chi \hat{\phi}$$

Drift kinetic susceptibility

$$\mathcal{L} = \left[\chi(k,\omega) - 1 - k^2\rho_s^2\right]\langle\hat{\phi}^2\rangle$$

$$N = \frac{\partial \chi}{\partial \omega}\langle\hat{\phi}^2\rangle$$

→ general expression, wave action density

$$v_{g,r} = -2k_r\rho_s^2 + \partial\chi/\partial k_r$$

- Spiral Waves

$\nabla^2\phi = 4\pi G\rho$

$\mathcal{L} = -\rho\phi - \frac{(\nabla\phi)^2}{8\pi G}$

but $\rho = \delta(z)\,\chi\,\phi$

~~effective~~ susceptibility

$\Rightarrow \left(\tilde{\sigma} = \chi\tilde{\phi}\right)$

$\Rightarrow \; -2|k|\,\tilde{\phi}_k = 4\pi G\,\tilde{\sigma}_k$

$$\left[1 + \frac{2\pi G\sigma_k}{|k|\tilde{\phi}_k}\right]\tilde{\phi}_k = 0$$

$$\epsilon_{eff} = 1 + \frac{2\pi G}{|k|}\chi(k,\omega)$$

$$\underbrace{\epsilon_{eff}}_{\text{generic dielectric form}} = 1 + \underbrace{\frac{2\pi G}{|k|}\chi(k,\omega_k)}_{\text{gives spiral dispersion relation}}$$

Now $\mathcal{L} = -\rho\phi - \frac{(\nabla\phi)^2}{8\pi G}$

$\Rightarrow$

$$\mathcal{L} = -\epsilon(k,\omega)\frac{|k|}{8\pi G}\overbrace{A}^{\text{wave amplitude}}{}^2$$

(i.e. $\langle(\nabla\phi)^2\rangle = -\langle\phi\,\nabla^2\phi\rangle$
$= +\langle\phi\,2|k|\,\phi\rangle$)

Thus,

$$\mathcal{I} = -\epsilon(k,\omega)\,\frac{|k|A^2}{8\pi G}$$

$$\epsilon(k,\omega) = 1 + \frac{2\pi G}{|k|}\chi(k,\omega)$$

$$\chi(k,\omega) = \delta\tilde{\Sigma}/\delta\tilde{\Phi}$$

$$\therefore \quad N = -\frac{\partial\epsilon}{\partial\omega}\,\frac{|k|}{8\pi G}A^2$$

$$= -\frac{2\pi G}{|k|}\left(\frac{\partial\chi}{\partial\omega}\right)\frac{|k|}{8\pi G}A^2$$

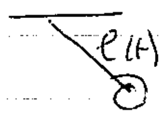
$$= -\left(\frac{\partial\chi}{\partial\omega}\right)\left(\frac{A^2}{4}\right)$$

Physics of Wave Action

→ wave action (conserved as consequence of phase symmetry)

→ adiabatic invariant

i.e. a/a'



$\ell(t)$

$\frac{\dot{\ell}}{\ell} < \omega$

so $E = 2\,\frac{1}{2} m\ell^2\dot{\theta}^2$

$= m\ell^2\omega^2\theta^2$

$E/\omega = m\ell^2\omega\,\theta^2$

$$\Rightarrow \frac{\partial}{\partial \ell}(\ell^2\omega)\,\overline{\theta^2} + (\ell^2\omega)\frac{\partial}{\partial \ell}\overline{\theta^2} \simeq 0$$

$$\frac{\Delta\overline{\theta^2}}{\overline{\theta^2}} = -\frac{\Delta(\ell^2\omega)}{\ell^2\omega}$$

→ requires $\omega > \frac{1}{k}\frac{dk}{dt}$

$\omega > \frac{1}{x}\frac{dx}{dt}$

→ obeys wave kinetic and continuity equations

i.e. $\frac{dN}{dt} = 0$, with $\frac{d\mathbf{x}}{dt} = \mathbf{V}_{gr} + \mathbf{V}$

$$\frac{d\mathbf{k}}{dt} = -\frac{\partial}{\partial \mathbf{x}}(\omega + \mathbf{k}\cdot\mathbf{V})$$

i.e. $$\boxed{\frac{\partial N}{\partial t} + (\underline{V}_{gr}+\underline{V})\cdot\underline{\nabla} N - \frac{\partial}{\partial \underline{x}}(\omega + \underline{k}\cdot\underline{V})\cdot\frac{\partial N}{\partial \underline{k}} = 0}$$

$$\frac{\partial N}{\partial t} + \underline{\nabla}\cdot\left((\underline{V}_{gr}+\underline{V})N\right) + \frac{\partial}{\partial \underline{k}}\left[-\frac{\partial}{\partial \underline{x}}(\omega+\underline{k}\cdot\underline{V})N\right] = 0$$

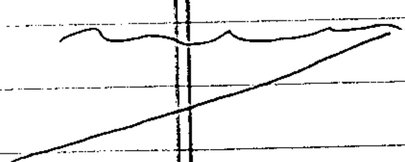
$\int d\underline{k}$ assuming N localized in $\underline{k}$,

$$\frac{\partial N}{\partial t} + \underline{\nabla}\cdot\left((\underline{V}_{gr}+\underline{V})N\right) = 0$$

if $\underline{V}$ irrelevant and $\partial\omega/\partial t = 0$,

$$\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla}\cdot(V_{gr}\mathcal{E}) = 0$$

→ can amplify waves o/s' beach



shallow water

$\omega^2 = g\,h(x)\,k^2 = \omega_0^2$
↳ excitation frequency

then have: $\omega = \text{const.}$ $\qquad \partial\omega/\partial t = 0 \Rightarrow d\omega/dt = 0$

$V_{gr}\,\mathcal{E} = \text{const.}$

Now, $V_{gr} = \sqrt{g h(x)}$

Beach ⟺ example of amplification by propagation

$\therefore$ $\sqrt{gh(x)}\ \mathcal{E}(x) = \text{const.}$

$\omega = k\sqrt{gh(x)} = \text{const.}$

$\mathcal{E} \uparrow$ with de-creasing depth
$k \uparrow$ ($\lambda \downarrow$) with decreasing depth
$\Big\rangle$ wave amplification (→ breaking)

For spirals: → Propagation

$$V_{gr} = -(\operatorname{sgn} k)\,(\pi G \sigma_0 - c_s^2 k)/(\omega - m\Omega)$$

$$\mathcal{E}_W = \frac{\omega_k\,\tilde{\omega}_k}{(\tilde{\omega}^2 - \kappa^2 - k^2 c_s^2)^2}\,\sigma_0\,\frac{|\tilde{\Phi}|^2}{2}$$

$$V_{gr}\,\mathcal{E} = -(\operatorname{sgn} k)\,\frac{(\pi G\sigma_0 - c_s^2 k)\,\omega_k\,\sigma_0}{(-2\pi G\sigma_0/|k|)^2}\,\frac{|\tilde{\Phi}|^2}{2}$$

$$= -\operatorname{sgn} k\,\frac{(\pi G\sigma_0 - c_s^2 k)}{(2\pi G\sigma_0)^2}\,\omega_k\,\sigma_0\,\frac{|\tilde{\Phi}_k|^2}{2}$$

and

$$\frac{dk_r}{dt} = -\frac{\partial}{\partial r}\left(\omega_k + m\Omega(r)\right)$$ → effective force

$$= -\frac{\partial}{\partial r}(\omega_k) - m\Omega'$$ ⇒ differential rotation drives spiral "wind up" (energy source)

$$V_{gr}\mathcal{E} = (\text{sgn}\, k_r)\frac{(c_s^2 k - \pi G \Sigma_0)}{(2\pi G\Sigma_0)^2}\omega_k \Sigma_0 |\tilde{\Phi}_k|^2$$

e.g.

∴ Long wave propagating toward co-rotation from $r < r_{co}$

$V_{gr} > 0$, $V_{gr}\mathcal{E} \sim$ above
$\sim k_r$ increasing

∴ $(\;) \uparrow$ so $|\tilde{\Phi}_k|^2 \uparrow$

⇒ some propagation induced amplification.

How ⇒ $\frac{dk_r}{dt} \cong -m\Omega'$ (wind up)

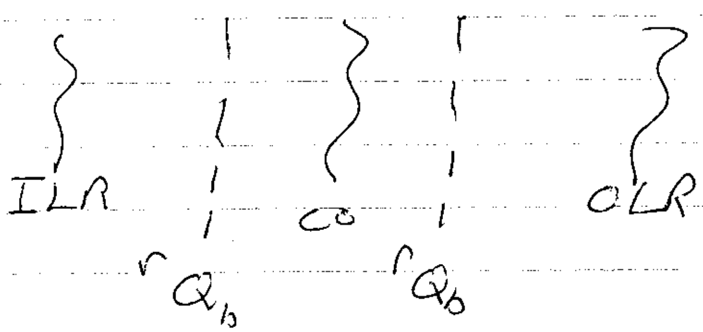
⇒ via $\langle \tilde{V}_r \tilde{V}_\theta \rangle$ driven work

⇒ still, simple propagation amplification is not promising.

∴ better to look at ⊕, ⊖ energy wave interaction at/across co-rotation.

→ Amplifying Spirals (Finally!)

→ have developed WKB/tight-winding theory of propagation on the Principal Range



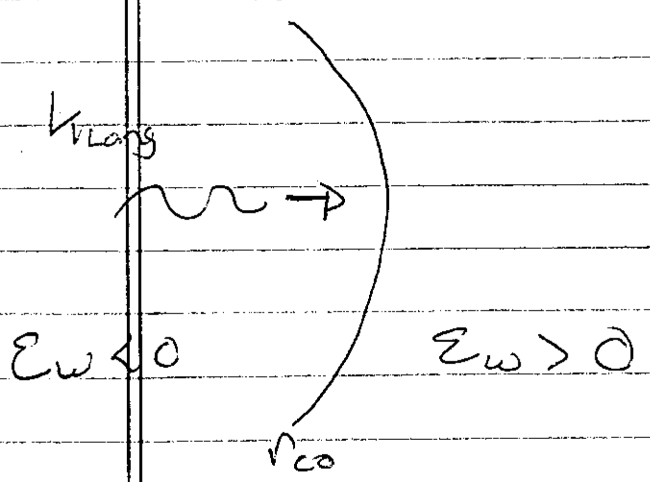
$r < r_{co} \rightarrow E_w < 0$
$r > r_{co} \rightarrow E_w > 0$

Long ⇒ always propagates toward r_{co}
short ⇒ propagates away from r_{co}

→ $Q_T > 1$ ⇒ need wave amplification mechanism

(*) – over-refraction / WASER
– SWING → Bar Instability ⟺ Global modes

→ "Over-Refraction"



consider $Q \cong 1$ (marginal) ⇒ ignore Q barrier

K_{Long} incident at r_{co} from $r_{ILR} < r < r_{co}$

Now:

r_{co}

→ have $\overrightarrow{k_{Long}}$ → transmitted

↑ incident

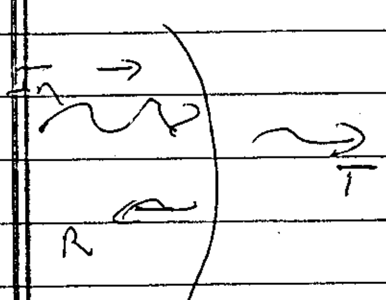
← reflected

but ⇒ $\begin{cases} \text{long waves propagate toward co-rotation} \\ \text{short waves propagate \underline{away from} co-rotation} \end{cases}$

∴ $k_{incident}$, $k_{transmitted}$ must be short waves ⇒ Long → short mode conversion must occur at r_{co}.

→ now, must conserve wave energy density flux across co-rotation

$$V_{gr}^{L}\,\mathcal{E}_{IN}^{L} = \underbrace{V_{gr}^{S}\,\mathcal{E}_{Refl}^{S}}_{r<r_{co}} + \underbrace{V_{gr}^{S}\,\mathcal{E}_{Tran.}^{S}}_{r>r_{co}}$$



Now, the point:

- if $\mathcal{E}_w > 0$ (all)

$\mathcal{E}^S_{Refl} < \mathcal{E}^L_{IN}$; necessarily, as energy lost to transmitted wave

- if $\mathcal{E}_w < 0$

$$V_{gr}^L \mathcal{E}^L_{IN} = V_{gr}^S \mathcal{E}^S_{Refl} + V_{gr}^S \mathcal{E}^S_{Trn}$$

$\ominus$ $\ominus$ $\ominus$

$$\therefore \frac{V_{gr}^L \mathcal{E}^L_{IN} - V_{gr}^S \mathcal{E}^S_{Trn}}{V_{gr}^S} = \mathcal{E}^S_{Refl}$$

$\ominus$ $\oplus$

$$\left(\frac{V_{gr}^L}{V_{gr}^S}\right) \mathcal{E}^L_{IN} - \mathcal{E}^S_{Trn} = \mathcal{E}^S_{Refl}$$

$\therefore$ $\mathcal{E}_w^{refl} < 0 \Rightarrow$ negative energy

$$\Rightarrow \boxed{|\mathcal{E}^S_{refl}| > |\mathcal{E}^L_{IN}|} \quad (V_g^L \sim V_g^S)$$

i.e. $\mathcal{E}_w = \left(\frac{\partial D}{\partial \omega}\right) |A|^2 \Rightarrow A$ must increase

"over-refraction"

∴ → reflected spiral amplitude amplified

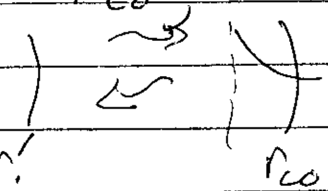
→ - negative energy wave crucial
- energy source is interaction with mean flow (work done vs differential rotation)
- contrast to linear instability
  ⇔ <u>not</u> exponential growth!

But:

→ What about Q barrier ??

- <u>if</u> → tunneling then weak amplification upon each scattering

  → if can reflect from some $r < r_{co}$ can build up



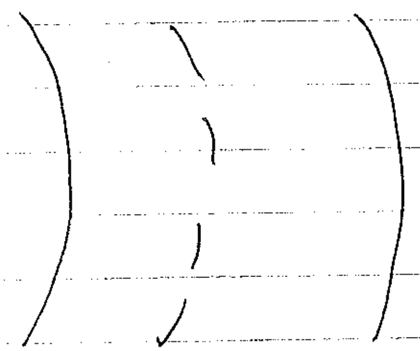
oscillator

⇒ WASER scheme

obviously, $r_{ILR} < r' < r_Q$

$\Rightarrow$ WASER (Wave Amplification by Stimulated Emission of Radiation)

$\rightarrow$ now, if $Q > 1 \Rightarrow \exists$ finite Q barrier



r_{Q-} r_{cc} r_{Q+}

$\Leftrightarrow$ waves must tunnel thru Q barrier to interact

$\rightarrow$ transmitted, over-refraction $O(\epsilon)$

i.e.

$$\mathcal{E}_0 \quad \mathcal{E}_R \quad \mathcal{E}_T$$

before: $-1 = -2 + 1$

$Q = 1 + \epsilon_0$

incident ($\mathcal{E}_w < 0$)

here: $-1 = \underbrace{-(1+\epsilon)}_{\text{over-refracted}} \underbrace{+\epsilon}_{\text{transmitted}}$

$\epsilon \equiv$ tunneling factor

→ even with Q-barrier, can increase amplitude of negative energy wave / short spiral, albeit increase small in tunnelling parameter

$$\epsilon \sim \exp\left[-2\int_{r_{Q-}}^{r_{co}} dr' |k_r|\right]$$

→ now, how build up wave (standing)

⇒ Find an inner reflection point!

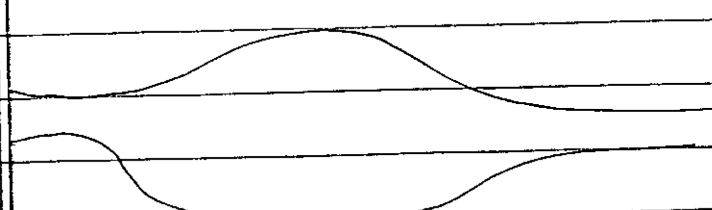
recall:

→ $V_{gr} = \mathrm{sgn}\, k_r \, (\pi G \Sigma_0 - k_r c_s^2)/\omega$

$$k_r > \frac{\pi G \Sigma_0}{c_s^2} \equiv k_T$$ → short wave

i.e. acoustic term dominates.

→ inner bulge



$\left|\frac{d\Sigma}{dr}\right|$ rapidly increases

$\therefore$ $v_{gr} \to 0$ when

$\pi \, \Sigma_0(r) = k_r c_s^2$ → inner reflection point

⇒ defines r_{IRP}

need $r_{ILR} < r_{IRP} < r_{Q_-}$ (not universal)
(or r_{ILB})

→ at r_{IRP}, inward propagating short wave converts to outward propagating long wave.

$\therefore$ restarts the cycle!

i.e. WASER Feedback loop: ⇔ standing spiral wave amplification mech.

i) – long wave incident on $r_{Q_-} < r < r_{co}$ from $r < r_{co}$

ii) – over-reflected short wave back scattered to $r < r_{co}$, propagating away from r_{co}, but amplified $|A|^2/|A_0|^2 \sim 1+G$

iii.) – back scattered short wave comes to r_{IRP}, due bulge, $V_{gr} \to 0$, so short wave scattered as long wave propagating toward r_{co}

iv.) – return to i.)

→ amount amplification?
→ NL saturation?

Finally, really no understanding of:

→ ammount of amplification (though see Lin, Mark, Lau papers – cited in Bertin and Lin)

→ nonlinear saturation → note that finite amplitude will alter $\Sigma(r)$ profile and also heat ⇒ feedback on Q, as well.

⇒ for deeper understanding:
Need examine Lindblad resonances and angular momentum exchange there!

⇒ Kinetics of spirals!